ON THE STRUCTURE OF BOREL STABLE ABELIAN SUBALGEBRAS IN INFINITESIMAL SYMMETRIC SPACES

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Abstract. Let $\mathfrak{g} = \mathfrak{g}^{\bar{0}} \oplus \mathfrak{g}^{\bar{1}}$ be a $\mathbb{Z}_2$-graded Lie algebra. We study the posets of abelian subalgebras of $\mathfrak{g}^{\bar{1}}$ which are stable w.r.t. a Borel subalgebra of $\mathfrak{g}^{\bar{0}}$. In particular, we find out a natural parametrization of maximal elements and dimension formulas for them. We recover as special cases several results of Kostant, Panyushev, Suter.

Contents

1. Introduction

Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra. Let σ be an involution of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{g}^{\bar{0}} \oplus \mathfrak{g}^{\bar{1}}$ be the corresponding eigenspace decomposition. Fix a Borel subalgebra $\mathfrak{b}^{\bar{0}}$ of the reductive Lie algebra $\mathfrak{g}^{\bar{0}}$. In this paper we deal with the following problem: *parametrize the maximal abelian $\mathfrak{b}^{\bar{0}}$-stable subalgebras of $\mathfrak{g}^{\bar{1}}$ and find formulas for their dimension.*

This kind of problem has ancient roots. A prototypical version of it is Schur's theorem [18], stating that there exist at most $\lfloor \frac{N^2}{4} \rfloor + 1$ linearly independent commuting matrices in $gl(N)$. To make a long story short, developments related to Schur's result (whose proof had been simplified by Jacobson [7] in the 50's) can be summed up as follows.

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1945: Malcev [13] found the maximal dimension of an abelian subalgebra of any simple $\mathfrak{g}$.	
1965: Kostant [9] found a connection between the eigenvalues of a Casimir of $\mathfrak{g}$ and the commutative subalgebras of $\mathfrak{g}$.	2001: Panyushev [15] generalized Kostant's results to the graded setting.
2000: Peterson's Abelian ideals theorem (cf. [11]): the abelian ideals of a Borel subalgebra of $\mathfrak{g}$ are $2^{rk(\mathfrak{g})}$.	2004: Cellini-Möseneder-Papi found a uniform enumeration of $\mathfrak{b}^{\bar{0}}$-stable abelian subalgebras of $\mathfrak{g}^{\bar{1}}$ (cf. [3]).
2003: Panuyshev [17] found a natural bijection between maximal abelian ideals of a Borel subalgebra of a simple Lie algebra and long simple roots.	
2004: Suter [19] gave a conceptual explanation of Malcev's result, providing a uniform formula for the dimension of maximal abelian ideals of a simple Lie algebra.	

In these terms, solving our initial problems means filling in the missing slots in the right column. Indeed, what links all these problems is their interpretation in terms of $\widehat{\mathfrak{u}}$-cohomology, $\widehat{\mathfrak{u}}$ being the nilpotent radical of the parabolic subalgebra $(\mathbb{C}[t] \otimes \mathfrak{g}) \cap \widehat{L}(\mathfrak{g}, \sigma)$ in the affine Kac-Moody algebra $\widehat{L}(\mathfrak{g}, \sigma)$. This remark is at the basis of Kostant's paper [12], and it is generalized to the graded setting in [14]. In the latter paper it is shown that combining Garland-Lepowsky theorem on $\widehat{\mathfrak{u}}$-cohomology with the relationships between the Laplacian associated to the standard Eilenberg-Chevalley boundary and the Casimir elements of $\widehat{L}(\mathfrak{g}, \sigma)$ and $\mathfrak{g}^{\bar{0}}$, it is possible to prove the following results, which motivate and give applications to our initial problem.

Given a commutative subalgebra $\mathfrak{a}$ of $\mathfrak{g}^{\bar{1}}$ with basis $v_1, \dots, v_k$, consider the vector $v_{\mathfrak{a}} = v_1 \wedge \cdots \wedge v_k \in \Lambda^k \mathfrak{g}^{\bar{1}}$ and let A_k be the span of the $v_{\mathfrak{a}}$'s when $\mathfrak{a}$ ranges over the k-dimensional commutative subalgebras of $\mathfrak{g}^{\bar{1}}$. Let finally m_k be the maximal eigenvalue of the Casimir element of $\mathfrak{g}^{\bar{0}}$ w.r.t. the Killing form of $\mathfrak{g}$ on $\Lambda^k \mathfrak{g}^{\bar{1}}$ and M_k the eigenspace of eigenvalue $k/2$.

Theorem (Kostant, Panyushev).

(1) $m_k \leq k/2$;
(2) $m_k = k/2$ *if and only if* $A_k \neq \emptyset$. *In such a case* $A_k = M_k$;
(3) $A = \sum_k A_k$ *is a multiplicity free* $\mathfrak{g}^{\bar{0}}$*-module whose irreducible pieces are indexed by the* $\mathfrak{b}^{\bar{0}}$*-stable abelian subalgebras of* $\mathfrak{g}^{\bar{1}}$.

Another result which is naturally explained by the cohomological approach is Peterson's theorem quoted above. This theorem admits an interpretation in terms of the geometry of alcoves, which we presently explain. Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$, which we temporarily assume to be simple. An abelian ideal $\mathfrak{i}$ of $\mathfrak{b}$, being stable w.r.t. the Cartan component of $\mathfrak{b}$, is a sum of root subspaces relative to a dual order ideal A of positive roots of $\mathfrak{g}$. Peterson's trick consists in considering the set of positive *affine* roots $-A + \delta$, δ being the fundamental imaginary root of $\widehat{L}(\mathfrak{g}, \sigma)$. It's easy to check that this set is biconvex, hence is a set of generalized

inversions of an element $w \in \widehat{W}$, the Weyl group of $\widehat{L}(\mathfrak{g},\sigma)$ (see Subsection 2.2). Peterson calls *minuscule* an element $w \in \widehat{W}$ associated, according to the above procedure, to an abelian ideal. It is shown in [1] that w is minuscule if and only if $wC_1 \subset 2C_1$, C_1 being the fundamental alcove, i.e. a fundamental domain for the affine action of $\widehat{W}$ on $(\mathfrak{h}_0)^*_{\mathbb{R}}$. This fact explains the enumerative result. It can be rephrased by saying that there exists a suitable simplex in $(\mathfrak{h}_0)^*_{\mathbb{R}}$ paved by the abelian ideals. The graded generalization found in [3], though much more complicated, is in the same spirit: the $\mathfrak{b}^{\bar{0}}$-stable abelian subalgebras of $\mathfrak{g}^{\bar{1}}$ are indexed by alcoves in a polytope D_σ, of which explicit equations are provided. This result will be recalled and refined in Section 3, and is the starting point for our investigation of maximal $\mathfrak{b}^{\bar{0}}$-stable abelian subalgebras of $\mathfrak{g}^{\bar{1}}$. Let $\mathcal{W}^{ab}_\sigma$ be the subset of $\widehat{W}$ formed by the elements indexing the alcoves of D_σ. We locate a special subset $\mathcal{M}_\sigma$ (see (3.3)) of bounding walls, with the property that if w is maximal (i.e., the corresponding $\mathfrak{b}^{\bar{0}}$-stable abelian subalgebra is), then $w(C_1)$ has a face on $\mathcal{M}_\sigma$. We are therefore reduced to study the posets

$$\mathcal{I}_{\alpha,\mu} = \{w \in \mathcal{W}^{ab}_\sigma \mid w(\alpha) = \mu\},$$

where α is a simple root of $\widehat{L}(\mathfrak{g},\sigma)$ and $\mu \in \mathcal{M}_\sigma$. This is done according to the following steps.

- We provide a criterion for $\mathcal{I}_{\alpha,\mu}$ to be non empty. Moreover we show that $\mathcal{I}_{\alpha,\mu}$, if non empty, has minimum (Theorem 4.10).
- We determine the poset structure of $\mathcal{I}_{\alpha,\mu}$, by relating it to a quotient of the subgroup $\widehat{W}_\alpha$ of $\widehat{W}$ generated by the simple reflections orthogonal to α by a reflection subgroup $\widehat{W}'_\alpha$ (Theorem 5.6).
- We look at intersections among the posets $\mathcal{I}_{\alpha,\mu}$, and we find necessary and sufficient conditions in order that the intersection of two such posets is non-void.
- We study maximal elements in $\mathcal{I}_{\alpha,\mu}$. We show that when $\widehat{W}'_\alpha$ is not standard parabolic, maximal elements appear in pairs of $\mathcal{I}_{\alpha,\mu}$'s: if w is maximal in $\mathcal{I}_{\alpha,\mu}$, then there exist a unique simple root β and a unique wall $\mu' \in \mathcal{M}_\sigma$ such that w is also maximal in $\mathcal{I}_{\beta,\mu'}$ (Lemma 7.4).
- We determine which maximal elements in $\mathcal{I}_{\alpha,\mu}$ are indeed maximal in $\mathcal{W}^{ab}_\sigma$ (Propositions 7.1, 7.2).

We finally provide a complete parametrization of maximal abelian $\mathfrak{b}^{\bar{0}}$-stable subalgebras (Theorem 7.3) and uniform formulas for their dimension (Corollary 7.6). Our results specialize nicely to Panyushev's and Suter's theorems quoted above (see Remark 7.1). But it is worthwhile to note that new phenomena appear, like the presence of maximal subalgebras indexed by certain pairs of simple roots lying in different components of Δ_0. To illustrate this fact, we state here our result in the special case when $\mathfrak{g}^{\bar{0}}$ is semisimple and σ is of inner type. In this case, by Kac's theory, we can choose a set of simple roots Π for $\mathfrak{g}$ in such a way that there is a unique simple root $\tilde{\alpha} \in \Pi$ such that $\sigma(x) = -x$ for $x \in \mathfrak{g}_{\tilde{\alpha}}$ and $\sigma(x) = x$ for $x \in \mathfrak{g}_\beta$ with $\beta \in \Pi \setminus \{\tilde{\alpha}\}$. Let $\Delta_0^+ = \coprod_{i=1}^r \Delta^+(\Sigma_i)$ be the decomposition of the (positive) root system of $\mathfrak{g}^{\bar{0}}$ into irreducible subsystems. Set $\mu_i = \delta - \theta_{\Sigma_i}$, $1 \le i \le r$, θ_{Σ_i} being the highest root of $\Delta(\Sigma_i)$.

Theorem. *In the above setting, the maximal $\mathfrak{b}^{\bar{0}}$-stable abelian subalgebras of $\mathfrak{g}^{\bar{1}}$ are:*

- $\max \mathcal{I}_{\alpha,\mu_i}$, $1 \le i \le r$, *α being a long simple root in* $\Gamma(\Sigma_i)$ *(see Def. 4.1) if θ_{Σ_i} is long or in Σ_i if θ_{Σ_i} is short;*
- $\max \mathcal{I}_{\tilde\alpha,\tilde\alpha+\delta}$*;*
- $\max\left(\mathcal{I}_{\alpha,\mu_i} \cap \mathcal{I}_{\beta,\mu_j}\right)$, $1 \le i < j \le r, \alpha \in \Sigma_j, \beta \in \Sigma_i$ *being long simple roots.*

The dimensions of these maximal subalgebras are given by formulas (7.4)*,* (7.5)*,* (7.6)*.*

2. Setup

2.1. **Twisted loop algebra and automorphisms.** Let $\mathfrak{g}, \sigma$ be as in the Introduction. We assume that σ is indecomposable, i.e. $\mathfrak{g}$ has no nontrivial σ-invariant ideals. Let $(\cdot,\cdot)$ be the Killing form of $\mathfrak{g}$. For $j \in \mathbb{Z}$ set $\bar j = j + 2\mathbb{Z}$, and let $\mathfrak{g}^{\bar j} = \{X \in \mathfrak{g} \mid \sigma(X) = (-1)^j X\}$, so that we have $\mathfrak{g} = \mathfrak{g}^{\bar 0} \oplus \mathfrak{g}^{\bar 1}$. We let $\widehat{L}(\mathfrak{g},\sigma)$ be the affine Kac-Moody Lie algebra associated to σ in [10, Section 8.2]. Let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{g}^{\bar 0}$. As shown in [10, Chapter 8], $\mathfrak{h}_0$ contains a regular element h_{reg} of $\mathfrak{g}$. In particular the centralizer $Cent(\mathfrak{h}_0)$ of $\mathfrak{h}_0$ in $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$ and h_{reg} defines a set of positive roots in the set of roots of $(\mathfrak{g}, Cent(\mathfrak{h}_0))$ and a set Δ_0^+ of positive roots in the set Δ_0 of roots for $(\mathfrak{g}^{\bar 0}, \mathfrak{h}_0)$. Since σ fixes h_{reg}, we see that the action of σ on the positive roots defines, once Chevalley generators are fixed, a diagram automorphism η of $\mathfrak{g}$ that, clearly, fixes $\mathfrak{h}_0$. Set, using the notation of [10], $\widehat{\mathfrak{h}} = \mathfrak{h}_0 \oplus \mathbb{C}K \oplus \mathbb{C}d$. Recall that d is the element of $\widehat{L}(\mathfrak{g},\sigma)$ acting on $\widehat{L}(\mathfrak{g},\sigma) \cap (\mathbb{C}[t,t^{-1}] \otimes \mathfrak{g})$ as $t\frac{d}{dt}$, while K is a central element. Define $\delta' \in \widehat{\mathfrak{h}}^*$ by setting $\delta'(d) = 1$ and $\delta'(\mathfrak{h}_0) = \delta'(K) = 0$ and let $\lambda \mapsto \overline{\lambda}$ be the restriction map $\widehat{\mathfrak{h}} \to \mathfrak{h}_0$. There is a unique extension, still denoted by $(\cdot,\cdot)$, of the Killing form of $\mathfrak{g}$ to a nondegenerate symmetric bilinear invariant form on $\widehat{L}(\mathfrak{g},\sigma)$. Let $\nu : \widehat{\mathfrak{h}} \to \widehat{\mathfrak{h}}^*$ be the isomorphism induced by the form $(\cdot,\cdot)$, and denote again by $(\cdot,\cdot)$ the form induced on $\widehat{\mathfrak{h}}^*$. One has $(\delta',\delta') = (\delta',\mathfrak{h}_0^*) = 0$.

We let $\widehat{\Delta}$ be the set of $\widehat{\mathfrak{h}}$-roots of $\widehat{L}(\mathfrak{g},\sigma)$. We can choose as set of positive roots $\widehat{\Delta}^+ = \Delta_0^+ \cup \{\alpha \in \widehat{\Delta} \mid \alpha(d) > 0\}$. We let $\widehat{\Pi} = \{\alpha_0, \dots, \alpha_n\}$ be the corresponding set of simple roots. It is known that n is the rank of $\mathfrak{g}^{\bar 0}$. Recall that any $\widehat{L}(\mathfrak{g},\sigma)$ is a Kac-Moody Lie algebra $\mathfrak{g}(A)$ defined by generator and relations starting from a generalized Cartan matrix A of affine type. These matrices are classified by means of Dynkin diagrams listed in [10].

Following [10, Chapter 8], we can assume that σ is the automorphism of type $(\eta; s_0, \dots, s_n)$, where η is the automorphism of the diagram defined above. Note that, since σ is an involution, $\eta^2 = Id$. We do not assume here that $\mathfrak{g}$ is simple, but, as explained in [8], most arguments given in [10] can be safely extended to the setting where $\mathfrak{g}$ is semisimple but not simple. This latter case, i.e. $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}$, $\mathfrak{k}$ a simple Lie algebra, σ the flip, will be referred to as the *adjoint case*. Recall that, if $a_0, \dots, a_n$ are the labels of the Dynkin diagram of $\widehat{L}(\mathfrak{g},\sigma)$ and k is the order of η, then $k(\sum_{i=0}^n s_i a_i) = 2$. Recall also that $s_0, \dots, s_n$ are relatively prime so we must have that $s_i \in \{0,1\}$ and $s_i = 0$ for all but at most two indices. The case in which we have two indices equal to 1 will be referred to as the *hermitian case* (indeed $\mathfrak{g}/\mathfrak{g}^{\bar 0}$ is an infinitesimal hermitian symmetric space). Since σ is the automorphism of type $(\eta; s_0, \dots, s_n)$, we can write $\alpha_i = s_i\delta' + \overline{\alpha_i}$ and the set $\Pi_0 = \{\alpha_i \mid s_i = 0\}$ is the set of simple roots for $\mathfrak{g}^{\bar 0}$ corresponding to Δ_0^+. Set also $\Pi_1 = \widehat{\Pi} \setminus \Pi_0$.

Introduce $\delta = \sum_{i=0}^n a_i\alpha_i$ and note that $\delta = (\sum_{i=0}^n a_i s_i)\delta' = \frac{2}{k}\delta'$. Set also $\alpha_i^\vee = \frac{2}{(\alpha_i,\alpha_i)}\nu^{-1}(\alpha_i)$ and let $\{a_0^\vee,\dots,a_n^\vee\}$ be the labels of the dual Dynkin diagram of $\widehat{L}(\mathfrak{g},\sigma)$.

We assume that K is the canonical central element [10, 6.2], $K = \sum_{i=0}^n a_i^\vee\alpha_i^\vee$. If we number the Dynkin diagrams as in [10, Tables Aff1, Aff2, Aff 3] then, by Sections 6.1, 6.2, 6.4 of [10],

$$K = \frac{2a_0}{\|\delta - a_0\alpha_0\|}\nu^{-1}(\delta). \tag{2.1}$$

Set finally $\mathbf{g} = \sum_{i=0}^n a_i^\vee$. This number is called the dual Coxeter number of $\widehat{L}(\mathfrak{g},\sigma)$.

We let $\widehat{W}$ be the Weyl group of $\widehat{L}(\mathfrak{g},\sigma)$. Set $(\mathfrak{h}_0)_\mathbb{R} = \oplus_{\alpha\in\Pi}\mathbb{R}\alpha^\vee$ and $\widehat{\mathfrak{h}}_\mathbb{R} = \mathbb{R}d \oplus \mathbb{R}K \oplus (\mathfrak{h}_0)_\mathbb{R}$. Set

$$C_1 = \{h \in (\mathfrak{h}_0)_\mathbb{R} \mid \overline{\alpha}_i(h) \geq -s_i,\ i = 0,\dots,n\} \tag{2.2}$$

be the fundamental alcove of $\widehat{W}$.

2.2. **Combinatorics of inversion sets.** For $w \in \widehat{W}$, we set

$$N(w) = \{\alpha \in \widehat{\Delta}^+ \mid w^{-1}(\alpha) \in -\widehat{\Delta}^+\}.$$

If α is a real root in $\widehat{\Delta}^+$, we let s_α denote the reflection in α. If α_i is a simple root we set $s_i = s_{\alpha_i}$.

The following facts are well-known. More details and references can be found in [2]. We will often use these properties in the rest of the paper without further notice.

(1) $N(w_1) = N(w_2) \implies w_1 = w_2$.

(2) if $w = s_{i_1}\cdots s_{i_k}$ is a reduced expression for w, then

$$N(w) = \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}),\dots, s_{i_1}\cdots s_{i_{k-1}}(\alpha_{i_k})\};$$

if moreover $\beta_h = s_{i_1}\cdots s_{i_{h-1}}(\alpha_{i_h})$, $1 \leq h \leq k$, then

$$w = s_{\beta_k}s_{\beta_{k-1}}\cdots s_{\beta_1}. \tag{2.3}$$

(3) $N(w)$ is biconvex, i.e. both $N(w)$ and $\widehat{\Delta}^+ \setminus N(w)$ are closed under root addition. Conversely, if $\widehat{\Delta}^+$ has no irreducible components of type $A_1^{(1)}$ and L is a finite subset of real roots which is biconvex, then there exists $w \in \widehat{W}$ such that $L = N(w)$.

(4) Denote by $\leq$ the weak left Bruhat order: $w_1 \leq w_2$ if there exists a reduced expression for w_1 which is an initial segment of a reduced expression for w_2). Then

$$w_1 < w_2 \iff N(w_1) \subset N(w_2).$$

(5) Set $N^\pm(w) = N(w) \cup -N(w)$. Then $N^\pm(w_1w_2) = N^\pm(w_1)\dot{+}w_1(N^\pm(w_2))$, where $\dot{+}$ denotes the symmetric difference. In particular, the following properties are equivalent:
   (a) $N(w_1w_2) = N(w_1) \cup w_1(N(w_2))$;
   (b) $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$;
   (c) $w_1(N(w_2)) \subset \widehat{\Delta}^+$.

We also introduce the sets of left and right descents for $w \in \widehat{W}$:

$$L(w) = \{\alpha \in \widehat{\Pi} \mid \ell(s_\alpha w) < \ell(w)\},$$
$$R(w) = \{\alpha \in \widehat{\Pi} \mid \ell(w s_\alpha) < \ell(w)\}.$$

We have that $L(w) = \widehat{\Pi} \cap N(w)$, $R(w) = \widehat{\Pi} \cap N(w^{-1})$.

2.3. **Conventions on root systems.**

2.3.1. We number affine Dynkin diagrams as in [10, Tables Aff1 and Aff2].

2.3.2. If $v \in \widehat{\mathfrak{h}}^*$, we set $v^\perp = \{x \in \widehat{\mathfrak{h}}^* \mid (x, v) = 0\}$.

2.3.3. If $S \subseteq \widehat{\Pi}$, we denote by $\Delta(S)$ (resp. $\Delta^+(S)$) the root system generated by S (resp. the set of positive roots corresponding to S). If $A \subseteq \widehat{\Delta}^+$ we denote by $W(A)$ the Weyl group generated (inside $\widehat{W}$) by the reflections in the elements of A.

We often identify subsets of the set of simple roots with their Dynkin diagram.

2.3.4. If R is a finite or affine root system and Π_R is a basis of simple roots, we write the expansion of a root $\gamma \in R$ w.r.t. Π_R as

$$\gamma = \sum_{\alpha \in \Pi_R} c_\alpha(\gamma)\gamma. \tag{2.4}$$

We also set, for $\alpha \in R$,

$$supp(\alpha) = \{\beta \in \Pi_R \mid c_\beta(\alpha) \neq 0\}.$$

2.3.5. If R is a finite irreducible root system and Π is a set of simple roots for R, we denote by θ_R (or by θ_Π) its highest root. Recall that the highest root and the highest short root are the only dominant weights belonging to R^+. We will use this remark in the following form:

$$\alpha \in R^+,\ \alpha \text{ long },\ (\alpha, \beta) \geq 0 \,\forall\, \beta \in R^+ \implies \alpha = \theta_R.$$

2.3.6. We recall the definition of dual Coxeter number g_R of a finite irreducible root system R. Write $\theta_R^\vee = \sum_{\alpha \in \Pi_R} c_{\alpha^\vee}(\theta^\vee)\alpha^\vee$ and set

$$g_R = 1 + \sum_{\alpha \in \Pi_R} c_{\alpha^\vee}(\theta^\vee). \tag{2.5}$$

2.4. **Reflection subgroups and coset representatives.** Let G be a finite or affine reflection group and let ℓ be the length function with respect to a fixed set of Coxeter generators S. Let R be the set of roots of G in the geometric representation, Π_R a system of simple roots for R, and R^+ the corresponding set of positive roots. Let G' be a subgroup of G generated by reflections, and R' be the set of roots $\alpha \in R$ such that $s_\alpha \in G'$, which is easily shown to be a root system. By [5],

$$\Pi_{R'} = \{\alpha \in R^+ \mid N(s_\alpha) \cap R' = \{\alpha\}\}$$

is a set of simple roots for R', whose associated set of positive roots is $R'^+ = R' \cap R^+$.

Given $g \in G$, we say that an element $w \in G'g$ is a minimal right coset representative if $\ell(w)$ is minimal among the lengths of elements of $G'g$. It follows from [5] by a standard argument that a coset $G'g$ has a unique minimal right coset representative w and this element is characterized by the following property:

$$w^{-1}(\alpha) \in R^+ \quad \text{for all } \alpha \in R'^+. \tag{2.6}$$

We will always choose as a coset representative for $G'g$ the minimal right coset representative and (with a slight abuse of notation) we denote by $G'\backslash G$ the set of all minimal right coset representatives. Thus the restriction of the weak order of G on $G'\backslash G$ induces a partial ordering on $G'\backslash G$. When saying *the poset* $G'\backslash G$, we shall always refer to this ordering.

2.4.1. If $\alpha \in R$ and G' is the stabilizer of α in G, then, for each $g \in G$, the minimal length representative of $G'g$ is the unique minimal length element that maps $g^{-1}\alpha$ to α. By formula 2.6, this element is characterized by the property

$$w^{-1}(\beta) \in R^+ \quad \text{for all } \beta \in R^+ \text{ orthogonal to } \alpha. \tag{2.7}$$

2.4.2. A reflection subgroup G' of G is standard parabolic when $\Pi_{R'} \subseteq \Pi_R$. In this case, if $g \in G$ and w is the minimal right coset representative of $G'g$, then $g = g'w$ with $g' \in G'$ and $\ell(g) = \ell(g') + \ell(w)$. In particular $N(g) \cap R' = N(g')$. Moreover, it is well known that g itself is the minimal representative of $G'g$ if and only if $L(g) \subseteq \Pi_R \setminus \Pi_{R'}$. Therefore $G'\backslash G = \{w \in G \mid L(w) \subseteq \Pi_R \setminus \Pi_{R'}\}$. If G is finite, the poset $G'\backslash G$ has a unique minimal and a unique maximal element. The identity of G clearly corresponds to the minimum of $G'\backslash G$. If w_0 is the longest element of G and w'_0 is the longest element of G', then we have that $N(w'_0 w_0) = R \setminus R'$. If $w \in G'\backslash G$, then $N(w) \subseteq R \setminus R'$; therefore $w'_0 w_0$ is the unique maximal element of $G'\backslash G$. Note that

$$\ell(w'_0 w_0) = |\Delta^+(R)| - |\Delta^+(R')|. \tag{2.8}$$

2.5. **Special elements in finite Weyl groups.** We sum up in the following statement the content of Propositions 7.1 and 7.2 from [2]. Attributions of the individual results are done there. The properties below will be used many times in the sequel.

Proposition 2.1. *Let R be a finite irreducible root system, W_R its Weyl group. Fix a positive system R^+ and let Π_R, θ_R be the corresponding set of simple root and highest root, respectively.*

1. *For any long root α there exists a unique element $y_\alpha \in W_R$ of minimal length such that $y(\alpha) = \theta_R$.*
2. *$L(y_\alpha) \subset \{\beta \in \Pi_R \mid (\beta, \theta_R) \neq 0\}$.*
3. *If conversely $v \in W_R$ is such that $v(\alpha) = \theta_R$ and $L(v) \subset \{\beta \in \Pi_R \mid (\beta, \theta_R) \neq 0\}$, then $v = y_\alpha$.*
4. *If $\alpha \in \Pi_R$, then $\ell(y_\alpha) = g_R - 2$, g_R being the dual Coxeter number of R.*
5. *If $\alpha \in \Pi_R$, and $\beta_1 + \beta_2 = \theta_R$, $\beta_1, \beta_2 \in R^+$, then exactly one element among β_1, β_2 belongs to $N(y_\alpha)$, and any element of $N(y_\alpha)$ arises in this way.*
6. *Conversely, if $y \in W_R$ is such that for any pair $\beta_1, \beta_2 \in R^+$ such that $\beta_1 + \beta_2 = \theta_R$ exactly one of β_1, β_2 belongs to $N(y)$ and $\theta_R \notin N(y)$, then there exists a long simple root β such that $y(\beta) = \theta_R$.*
7. *$N(y_\alpha^{-1}) = \{\beta \in R^+ \mid (\beta, \alpha^\vee) = -1\}$.*
8. *$\gamma \in R^+$, $(\gamma, \theta_R) = 0 \implies \gamma \notin N(y_\alpha)$.*

3. Borel stable abelian subalgebras and affine Weyl groups

Recall that Π_0 denotes the set of simple roots of $\mathfrak{g}^{\bar{0}}$ corresponding to Δ_0^+. In general Π_0 is disconnected and we write $\Sigma|\Pi_0$ to mean that Σ is a connected component

of Π_0. Clearly, the Weyl group W_0 of $\mathfrak{g}^{\bar{0}}$ is the direct product of the $W(\Sigma)$, $\Sigma|\Pi_0$. If θ_Σ is the highest root of $\Delta(\Sigma)$, set

$$\begin{aligned}
\widehat{\Delta}_0 &= \{\alpha + \mathbb{Z}k\delta \mid \alpha \in \Delta_0\} \cup \pm\mathbb{N}k\delta,\\
\widehat{\Pi}_0 &= \Pi_0 \cup \{k\delta - \theta_\Sigma \mid \Sigma|\Pi_0\},\\
\widehat{\Delta}_0^+ &= \Delta_0^+ \cup \{\alpha \in \widehat{\Delta}_0 \mid \alpha(d) > 0\}.
\end{aligned}$$

Denote by $\widehat{W_0}$ the Weyl group of $\widehat{\Delta}_0$. Let $\widehat{\Delta}_{re} = \widehat{W}\widehat{\Pi}$ be the set of real roots of $\widehat{L}(\mathfrak{g},\sigma)$. If $\lambda \in \mathfrak{h}_0^*$, then we let $\mathfrak{g}_\lambda \subset \mathfrak{g}$ be the corresponding weight space. We say that a real root α is noncompact if $\mathfrak{g}_{\overline{\alpha}} \subset \mathfrak{g}^{\bar{1}}$, compact if $\mathfrak{g}_{\overline{\alpha}} \subset \mathfrak{g}^{\bar{0}}$, and complex if it is neither compact nor noncompact. Note that, by the very definition of $\widehat{L}(\mathfrak{g},\sigma)$, if $\alpha \in \widehat{\Delta}_{re}$, then $k\delta + \alpha \in \widehat{\Delta}$, while, if $k = 2$, $\delta + \alpha \in \widehat{\Delta}$ if and only if α is complex. Clearly, if $\eta = Id$, then any real root is either compact or noncompact. It is shown in [4] that, if $\mathfrak{g}$ is simple and $\eta \neq Id$, then a real root α is either compact or noncompact if and only if α is a long root (i.e., $\|\alpha\|$ is largest among the possible root lengths). If $\mathfrak{g}$ is not simple, since σ is indecomposable, all the real roots are complex.

If $\alpha \in \widehat{\Delta}$, set (cf. (2.4))

$$ht_\sigma(\alpha) = \sum_{i=0}^{n} s_i c_{\alpha_i}(\alpha)$$

and, for $i \in \mathbb{Z}$,

$$\widehat{\Delta}_i = \{\alpha \in \widehat{\Delta} \mid ht_\sigma(\alpha) = i\}.$$

Remark 3.1. Since $\alpha_i = s_i\delta' + \overline{\alpha}_i$ (Section 2.1), for any $\alpha \in \widehat{\Delta}$, we have that $\alpha = ht_\sigma(\alpha)\delta' + \overline{\alpha}$. In particular, since $k\delta = 2\delta'$, $ht_\sigma(k\delta) = 2$. By definition, the roots θ_Σ, $\Sigma|\Pi_0$, are the maximal roots having σ-height equal to 0, with respect to the usual order $\leq$ on roots: $\alpha \leq \beta$ if and only if $\beta - \alpha$ is a sum of positive roots or zero. It follows that the roots $k\delta - \theta_\Sigma$ are the minimal roots having σ-height equal to 2. More generally, if $s \in \mathbb{Z}$, $\{sk\delta - \theta_\Sigma \mid \Sigma|\Pi_0\}$ is the set of minimal roots in $\widehat{\Delta}_{2s}$. Similarly, $\Pi_1 + sk\delta$ is the set of minimal roots in $\widehat{\Delta}_{2s+1}$.

Definition 3.1. *An element $w \in \widehat{W}$ is called σ-minuscule if $N(w) \subset \widehat{\Delta}_1$. We denote by $\mathcal{W}_\sigma^{ab}$ the set of σ-minuscule elements of $\widehat{W}$.*

We regard $\mathcal{W}_\sigma^{ab}$ as a poset under the weak Bruhat order.

Remark 3.2. Note that in the adjoint case $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}$, $\mathfrak{k}$ simple, w is σ-minuscule if and only if $N(w) \subset -\Delta_{\mathfrak{k}}^+ + \delta$, $\Delta_{\mathfrak{k}}^+$ being the set of positive roots of $\mathfrak{k}$. So we recover Peterson's notion of minuscule elements quoted in the Introduction.

Remark 3.3. It will be useful, from a notational point of view, to introduce the following generalization of the σ-height. Given $A \subseteq \widehat{\Pi}$ and $\gamma \in \widehat{\Delta}$, set

$$ht_A(\gamma) = \sum_{\alpha \in A} c_\alpha(\gamma).$$

In particular, the σ-height equals ht_{Π_1} and the usual height equals $ht_{\widehat{\Pi}}$. In these two cases we will keep using ht_σ, ht.

Let a be the squared length of a long root in $\widehat{\Delta}^+$. Define

$$\widehat{\Pi}_0^* = \Pi_0 \cup \left\{k\delta - \theta_\Sigma \mid a \leq 2\|\theta_\Sigma\|^2\right\}, \tag{3.1}$$

$$\Phi_\sigma = \widehat{\Pi}_0^* \cup \{\alpha + k\delta \mid \alpha \in \Pi_1, \alpha \text{ long and noncomplex}\} \tag{3.2}$$

Remark 3.4.
(1) It is immediate to see that $\widehat{\Pi}_0^* = \widehat{\Pi}_0$, unless $\widehat{L}(\mathfrak{g},\sigma)$ is of type $G_2^{(1)}$ or $A_2^{(2)}$. Indeed, in the latter cases there exists $\Sigma|\Pi_0$ such that $\frac{a}{\|\theta_\Sigma\|^2} = 3, 4$, respectively.
(2) When $|\Pi_1| = 2$, then both roots in Π_1 are long; moreover, for any $\Sigma|\Pi_0$, both roots in Π_1 are not orthogonal to Σ. This is most easily seen by a brief inspection of the untwisted Dynkin diagrams, recalling that, by Section 2.1, $k = 1$ and the labels of the roots in Π_1 in the Dynkin diagram of $\widehat{\Pi}$ are equal to 1. Anyway, we provide a uniform argument. Let $\Pi_1 = \{\alpha, \beta\}$: since $k = 1$ and $c_\alpha(\delta) = 1$, $\delta - \alpha$ is a root and belongs to $\Delta(\widehat{\Pi} \setminus \{\alpha\})$. Since the support of $\delta - \alpha$ is $\widehat{\Pi} \setminus \{\alpha\}$, we see that $\widehat{\Pi} \setminus \{\alpha\}$ is connected. We claim that $\delta - \alpha$ is the highest root $\Delta(\widehat{\Pi} \setminus \{\alpha\})$. Otherwise, if $\beta > \delta - \alpha$ and $\beta \in \Delta(\widehat{\Pi} \setminus \{\alpha\})$, then $\beta - \delta$ would be a root with positive coefficients in some simple root in $\widehat{\Pi} \setminus \{\alpha\}$ and coefficient -1 in α. In particular, we obtain that $\delta - \alpha$ is long with respect to $\Delta(\widehat{\Pi} \setminus \{\alpha\})$ and, since it has the same length as α, that both $\delta - \alpha$ and α are long. For proving the second claim, observe that $\Sigma \cup \{\beta\} \subseteq Supp(\delta - \alpha) = \widehat{\Pi} \setminus \{\alpha\}$ and the latter is connected. Hence β has to be nonorthogonal to Σ. Switching the role of α and β we get the second claim.

Consider the set

$$D_\sigma = \bigcup_{w \in W_{ab}^\sigma} wC_1.$$

(cf. (2.2)). If $\alpha \in \widehat{\Delta}$ then we let $H_\alpha^+ = \{h \in (\mathfrak{h}_0)_\mathbb{R} \mid \alpha(d + h) \geq 0\}$. The following result refines [3, Proposition 4.1].

Proposition 3.1.

$$D_\sigma = \bigcap_{\alpha \in \Phi_\sigma} H_\alpha^+.$$

Proof. By [3, Propositions 4.1 and 5.8] and by Remark 3.4 (2), we have that $D_\sigma = \bigcap_{\alpha \in \Phi'_\sigma} H_\alpha^+$, where $\Phi'_\sigma = \widehat{\Pi}_0 \cup \{\alpha + k\delta \mid \alpha \in \Pi_1, \alpha \text{ long and noncomplex}\}$. (Actually Propositions 4.1 and 5.8 of [3] cover only the cases when $\mathfrak{g}$ is simple, but the argument is easily extended to the adjoint case.) Therefore, we have only to prove that we can restrict from $\widehat{\Pi}_0$ to $\widehat{\Pi}_0^*$, i.e. that if Σ is a component of Π_0 such that $a > 2\|\theta_\Sigma\|^2$, then $\theta_\Sigma(x) \leq k$ for all $x \in D_\sigma$. By Remark 3.4 (1), $\widehat{\Pi}$ is of type $G_2^{(1)}$ or $A_2^{(2)}$, in particular Π_1 has a single element: set $\Pi_1 = \{\tilde{\alpha}\}$. Note that $\tilde{\alpha}$ is long. We proceed in steps.

(1) $\tilde{\alpha} + 3\theta_\Sigma \in \widehat{\Delta}^+$: this follows from $(\tilde{\alpha}, \theta_\Sigma^\vee) < -2$.
(2) $2\tilde{\alpha} + 3\theta_\Sigma \in \widehat{\Delta}^+_{re}$: indeed $(\tilde{\alpha}, \tilde{\alpha} + 3\theta_\Sigma) < 0$ and $\|2\tilde{\alpha} + 3\theta_\Sigma\| > 0$.
(3) $k\delta - 2\tilde{\alpha} - 3\theta_\Sigma \in \Delta_0^+$: relation $k\delta - 2\tilde{\alpha} - 3\theta_\Sigma \in \widehat{\Delta}$ follows from (2); it is also clear that it belongs to Δ_0. So it remains to show that it is positive. Indeed (1) implies $k\delta - \tilde{\alpha} - 3\theta_\Sigma \in \widehat{\Delta}$, and this root is positive since $c_{\tilde{\alpha}}(k\delta - \tilde{\alpha} - 3\theta_\Sigma) = 1$, hence $(k\delta - \tilde{\alpha} - 3\theta_\Sigma) - \tilde{\alpha} \in \widehat{\Delta}^+$.

Now we can conclude, since $(k\delta - 2\tilde{\alpha} - 3\theta_\Sigma)(x) \geq 0$ implies $\theta_\Sigma(x) \leq \frac{k}{3} - \frac{2}{3}(\tilde{\alpha}, x) \leq k$. $\square$

Remark 3.5. In the adjoint case $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}$, $\mathfrak{k}$ simple, D_σ is twice the fundamental alcove of the affine Weyl group of $\mathfrak{k}$.

We let $\mathcal{I}^\sigma_{ab}$ be the set of abelian subalgebras in $\mathfrak{g}^{\bar{1}}$ that are stable under the action of the Borel subalgebra $\mathfrak{b}^{\bar{0}}$ of $\mathfrak{g}^{\bar{0}}$ corresponding to Δ_0^+. Inclusion turns $\mathcal{I}^\sigma_{ab}$ into a poset.

Proposition 3.2. [3, Theorem 3.2] *Let $w \in \mathcal{W}^{ab}_\sigma$. Suppose $N(w) = \{\beta_1, \dots, \beta_k\}$. The map $\mathcal{W}^{ab}_\sigma \to \mathcal{I}^\sigma_{ab}$ defined by*

$$w \mapsto \bigoplus_{i=1}^{k} \mathfrak{g}^{\bar{1}}_{-\overline{\beta}_i}$$

is a poset isomophism.

Remark 3.6. The natural isomorphism of $\mathfrak{g}^{\bar{0}}$-modules $\mathfrak{g}^{\bar{1}} \cong t^{-1} \otimes \mathfrak{g}^{\bar{1}}$ maps the $\mathfrak{b}^{\bar{0}}$-stable abelian subspaces of $\mathfrak{g}^{\bar{1}}$ to $\mathfrak{b}^{\bar{0}}$-stable abelian subspaces of $\widehat{L}(\mathfrak{g}, \sigma)$. Through this isomorphism, the map of the above proposition associates to $w \in \mathcal{W}^{ab}_\sigma$ the $\mathfrak{b}^{\bar{0}}$-stable abelian subalgebra $\bigoplus_{i=1}^k \widehat{L}(\mathfrak{g}, \sigma)_{-\beta_i}$.

Set

$$\mathcal{M}_\sigma = \Phi_\sigma \backslash (\widehat{\Pi} \cap \Phi_\sigma). \tag{3.3}$$

Proposition 3.3. *If $w \in \mathcal{W}^{ab}_\sigma$ is maximal, then there is $\alpha \in \widehat{\Pi}$ and $\mu \in \mathcal{M}_\sigma$ such that $w(\alpha) = \mu$.*

Proof. By Proposition 3.1, we have that, if $\alpha \in \widehat{\Pi}$, $w(\alpha) \in \widehat{\Delta}^+$, then $ws_\alpha(C_1) \not\subset D_\sigma$, hence there exists $\mu \in \Phi_\sigma$ such that $ws_\alpha(C_1) \not\subset H^+_\mu$. It follows that $\mu \in N(ws_\alpha)$. Since $N(ws_\alpha) = N(w) \cup \{w(\alpha)\}$, we see that $w(\alpha) = \mu$. We need therefore to prove that there is a simple root α such that $w(\alpha) \in \widehat{\Delta}^+$ and $w(\alpha) \notin \Pi_0$.

Assume on the contrary that, if $\alpha \in \widehat{\Pi}$ and $w(\alpha) \in \widehat{\Delta}^+$, then $w(\alpha) \in \Pi_0$. Then, for all $\alpha \in \widehat{\Pi}$, $ht_\sigma(w(\alpha)) \leq 0$ and, hence, for all $\beta \in \widehat{\Delta}^+$, we have that $ht_\sigma(w(\beta)) \leq 0$. It follows that, for all $\beta \in \widehat{\Delta}^+$, if $w(\beta)$ is positive, then $w(\beta) \in \Delta_0$. Equivalently, $w(\widehat{\Delta}^+) \cap \widehat{\Delta}^+ \subseteq \Delta_0$. Hence, in particular, $w(\widehat{\Delta}^+) \setminus \widehat{\Delta}^+$ is infinite, but this is impossible, since $w(\widehat{\Delta}^+) \setminus \widehat{\Delta}^+ = -N(w)$. $\square$

4. The poset $\mathcal{I}_{\alpha,\mu}$ and its minimal elements

Given $\alpha \in \widehat{\Pi}$, $\mu \in \mathcal{M}_\sigma$, set

$$\mathcal{I}_{\alpha,\mu} = \{w \in \mathcal{W}^{ab}_\sigma \mid w(\alpha) = \mu\}.$$

In this Section we find necessary and sufficient conditions for the poset $\mathcal{I}_{\alpha,\mu}$ to be nonempty, and in such a case we show that it has minimum.

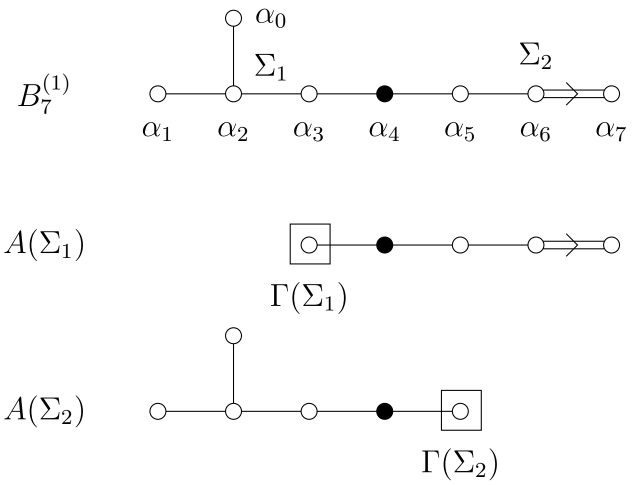
We consider first the case $\mu = k\delta - \theta_\Sigma$, with $\Sigma | \widehat{\Pi}_0$.

Definition 4.1. *Let $\Sigma|\Pi_0$, and consider the subgraph of $\widehat{\Pi}$ with $\{\alpha \in \widehat{\Pi} \mid (\alpha, \theta_\Sigma) \leq 0\}$ as set of vertices. We call $A(\Sigma)$ the union of the connected components of this subgraph which contain at least one root of Π_1. Moreover, we set*

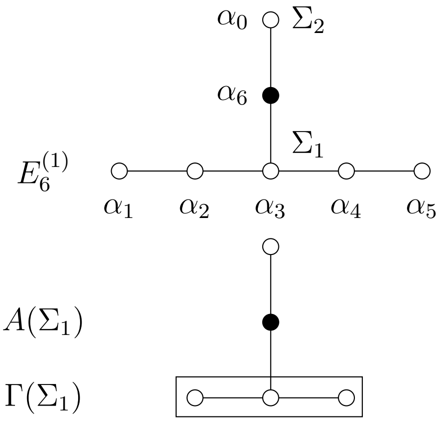
$$\Gamma(\Sigma) = A(\Sigma) \cap \Sigma.$$

Remark 4.1. If $|\Pi_1| = 1$ then, obviously, $A(\Sigma)$ is connected. If $|\Pi_1| = 2$ then a brief inspection shows that there is only one case when $A(\Sigma)$ is disconnected, namely when $\widehat{\Pi}$ is of type $C_n^{(1)}$. Note that in such a case Π_0 is connected and θ_{Π_0} is a short root.

Example 4.1. (1). Let $\widehat{L}(\mathfrak{g}, \sigma)$ be of type $B_n^{(1)}$ $(n \geq 5)$ and $\Pi_1 = \{\alpha_p\}$, $4 \leq p \leq n-1$. Then Π_0 has two components, say Σ_1, of type D_p, with simple roots $\{\alpha_i, \mid 0 \leq i \leq p-1\}$, and Σ_2 of type B_{n-p} and simple roots $\{\alpha_i, \mid p+1 \leq i \leq n\}$. We have $A(\Sigma_1) = \{\alpha_{p-1}, \dots, \alpha_n\}$, $\Gamma(\Sigma_1) = \{\alpha_{p-1}\}$, and $A(\Sigma_2) = \{\alpha_0, \dots, \alpha_{p+1}\}$, $\Gamma(\Sigma_2) = \{\alpha_{p+1}\}$. We illustrate this example in the case $n = 7, p = 4$.

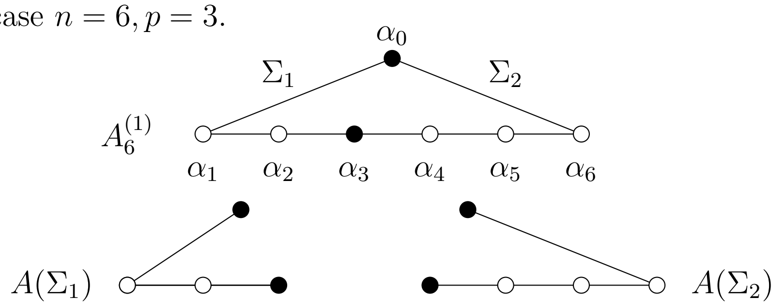


(2). Let $\widehat{L}(\mathfrak{g}, \sigma)$ be of type $E_6^{(1)}$ and $\Pi_1 = \{\alpha_6\}$. Then Π_0 has two components: Σ_1, of type A_5, with simple roots $\{\alpha_1, \dots, \alpha_5\}$, and $\Sigma_2 = \{\alpha_0\}$, of type A_1. We have $A(\Sigma_1) = \{\alpha_2, \alpha_3, \alpha_4, \alpha_6, \alpha_0\}$, $\Gamma(\Sigma_1) = \{\alpha_2, \alpha_3, \alpha_4\}$ and $A(\Sigma_2) = \widehat{\Pi} \setminus \{\alpha_0\}$, $\Gamma(\Sigma_2) = \emptyset$.



(3). Let $\widehat{L}(\mathfrak{g}, \sigma)$ be of type $A_n^{(1)}$, $(n > 2)$, and $\Pi_1 = \{\alpha_0, \alpha_p\}$, $1 < p < n$. Then Π_0 has two components: Σ_1, of type A_{p-1}, with simple roots $\{\alpha_i, \mid 1 \leq i \leq p-1\}$, and Σ_2 of type A_{n-p} and simple roots $\{\alpha_i, \mid p+1 \leq i \leq n\}$. We have and $A(\Sigma_1) =$

$\Sigma_2 \cup \Pi_1$, $A(\Sigma_2) = \Sigma_1 \cup \Pi_1$, and $\Gamma(\Sigma_i) = \emptyset$ for $i = 1, 2$. In the following picture we display the case $n = 6, p = 3$.



Remark 4.2. Assume that $\Sigma|\Pi_0$, $k\delta - \theta_\Sigma \in \mathcal{M}_\sigma$, $\alpha \in \Pi_1$, and set

$$r_\Sigma = -(\alpha, \theta_\Sigma^\vee).$$

By Remark 3.4 (2), r_Σ is independent from the choice of $\alpha \in \Pi_1$. Moreover, we see that $r_\Sigma = 1$ if and only if θ_Σ is long and non complex while, in the remaining cases, since we are assuming that $k\delta - \theta_\Sigma \in \widehat{\Pi}_0^*$, we have that $r_\Sigma = 2$. If $r_\Sigma = 2$, then, for $\alpha \in \Pi_1$, either $\|\alpha\| = 2\|\theta_\Sigma\|$, or $\overline{\alpha} = -\overline{\theta}_\Sigma$. The latter instance occurs in the adjoint case, so that $k = 2$ and θ_Σ is long and complex. In the first case, θ_Σ is a short root, and k may be 1 or 2. In fact, $k = 2$ and θ_Σ is complex, except in the following two cases: $\mathfrak{g}$ is of type B_n, $\Pi_1 = \{\alpha_{n-1}\}$ and $\theta_\Sigma = \alpha_n$ or $\mathfrak{g}$ is of type C_n, $\Pi_1 = \{\alpha_0, \alpha_n\}$, $\Sigma = \{\alpha_1, \dots, \alpha_{n-1}\}$.

From now on we will distinguish roots in two types, according to the following definition.

Definition 4.2. *We say that $\alpha \in \widehat{\Delta}_{re}^+$ is of type 1 if it is long and non complex and of type 2 otherwise.*

By the above remark, if $k\delta - \theta_\Sigma \in \mathcal{M}_\sigma$, its type is r_Σ.

Lemma 4.1. *Assume $\Sigma|\Pi_0$ and $k\delta - \theta_\Sigma \in \mathcal{M}_\sigma$. If $\frac{k}{r_\Sigma} \in \mathbb{Z}$, then $A(\Sigma)$ is connected, $\frac{k}{r_\Sigma}\delta - \theta_\Sigma$ is a root, and*

$$supp\left(\frac{k}{r_\Sigma}\delta - \theta_\Sigma\right) \subseteq A(\Sigma). \tag{4.1}$$

Proof. Note that $\frac{k}{r_\Sigma} \in \mathbb{Z}$ if and only if $r_\Sigma = 1$ or $k = r_\Sigma = 2$, in any case $\frac{k}{r_\Sigma} \in \{1, 2\}$. If $\frac{k}{r_\Sigma} = 2$, then $\frac{k}{r_\Sigma}\delta - \theta_\Sigma \in \Delta$ and, if $\frac{k}{r_\Sigma} = 1$ then, either $k = 1$ or $k = 2$ and θ_Σ is complex. In both cases, $\frac{k}{r_\Sigma}\delta - \theta_\Sigma \in \Delta$.

We now prove that $supp(\frac{k}{r_\Sigma}\delta - \theta_\Sigma) \subset A(\Sigma)$. Note that $\Pi_1 \subset supp(\frac{k}{r_\Sigma}\delta - \theta_\Sigma)$, hence we need only to prove that $\alpha \notin supp(\frac{k}{r_\Sigma}\delta - \theta_\Sigma)$ for any $\alpha \in \Sigma$ such that $(\alpha, \theta_\Sigma) > 0$. We next show that, for such an α, we have $c_\alpha(\frac{k}{r_\Sigma}\delta - \theta_\Sigma) = 0$. We have:

$$2\frac{r_\Sigma}{k} = -\sum_{\beta \in \Pi_1} c_\beta(\delta)(\beta, \theta_\Sigma^\vee) = \sum_{\substack{\beta \in \Sigma \\ (\beta, \theta_\Sigma) > 0}} c_\beta(\delta)(\beta, \theta_\Sigma^\vee).$$

The first equality follows by the definition of r_Σ, and the second by the relation $(\delta, \theta_\Sigma) = 0$. If there is only one root $\alpha \in \Sigma$ such that $(\alpha, \theta_\Sigma) > 0$, we obtain that

$$\frac{k}{r_\Sigma} c_\alpha(\delta)(\alpha, \theta_\Sigma^\vee) = 2 = c_\alpha(\theta_\Sigma)(\alpha, \theta_\Sigma^\vee),$$

hence $c_\alpha(\frac{k}{r_\Sigma}\delta - \theta_\Sigma) = 0$. If there is more than one root in Σ not orthogonal to θ_Σ then $\sum\limits_{\substack{\alpha\in\Sigma\\(\alpha,\theta_\Sigma)>0}} c_\alpha(\theta_\Sigma)(\alpha,\theta_\Sigma^\vee) = 2$, hence $(\alpha,\theta_\Sigma) = c_\alpha(\theta_\Sigma) = 1$ for all $\alpha \in \Sigma$ not orthogonal to θ_Σ.

Since $\frac{k}{r_\Sigma}\sum\limits_{\substack{\alpha\in\Sigma\\(\alpha,\theta_\Sigma)>0}} c_\alpha(\delta) = 2$, $\frac{k}{r_\Sigma} \in \mathbb{Z}$, and $c_\alpha(\delta) > 0$ for all $\alpha \in \widehat{\Pi}$, we obtain $\frac{k}{r_\Sigma}c_\alpha(\delta) = 1$ and again we have $c_\alpha(\frac{k}{r_\Sigma}\delta - \theta_\Sigma) = 0$, as desired. □

Note that, if θ_Σ is of type 1 or $k = 2$, then $\frac{k}{r_\Sigma} \in \mathbb{Z}$. In particular $A(\Sigma)$ is connected.

Proposition 4.2. *Assume $\Sigma|\Pi_0$ and $k\delta-\theta_\Sigma \in \mathcal{M}_\sigma$. If θ_Σ is of type 1, then $k\delta-\theta_\Sigma$ is the highest root of $\Delta(A(\Sigma))$. If $k = 2$ and θ_Σ is of type 2, then $\delta - \theta_\Sigma$ is either the highest root of $\Delta(A(\Sigma))$, or its highest short root.*

Proof. Our assumptions imply in any case that $\frac{k}{r_\Sigma} \in \mathbb{Z}$. By (4.1) we have that $\frac{k}{r_\Sigma}\delta - \theta_\Sigma \in \Delta(A(\Sigma))$. By the definition of $A(\Sigma)$, $\frac{k}{r_\Sigma}\delta - \theta_\Sigma$ is a dominant root in $\Delta(A(\Sigma))$, therefore, since $\Delta(A(\Sigma))$ is a finite root system, we obtain that it is either the highest root of $\Delta(A(\Sigma))$ or its highest short root. If θ_Σ is of type 1, then it is a long root, so, since $r_\Sigma = 1$, $k\delta - \theta_\Sigma$ is the highest root of $\Delta(A(\Sigma))$. If θ_Σ is of type 2, then $r_\Sigma = 2$, hence $\frac{k}{r_\Sigma} = 1$. In this case, θ_Σ may be short or long, and $\delta - \theta_\Sigma$ is the highest short or long root of $\Delta(A(\Sigma))$, according to its length. □

Lemma 4.3. *Assume $\Sigma|\Pi_0$, $k\delta - \theta_\Sigma \in \mathcal{M}_\sigma$ and θ_Σ of type 2. Let s be the element of minimal length in $\widehat{W}$ such that $s(\theta_\Sigma) = k\delta - \theta_\Sigma$. Then $s \in W(A(\Sigma))$ and is an involution. Moreover,*

$$N(s) = \{\beta \in \widehat{\Delta}_1^+ \mid (\beta, \theta_\Sigma^\vee) = -2\},$$

in particular, $s \in \mathcal{W}_\sigma^{ab}$.

Proof. First we assume $k = 2$. We claim that in this case $s = s_{\delta-\theta_\Sigma}$, which directly implies that it is an involution and, by Proposition 4.2, that it belongs to $W(A(\Sigma))$. It is immediate that $s_{\delta-\theta_\Sigma}(\theta_\Sigma) = 2\delta - \theta_\Sigma$. Moreover, for each $\alpha \in \widehat{\Delta}^+$ which is orthogonal to θ_Σ we have $s_{\delta-\theta_\Sigma}(\alpha) = \alpha \in \widehat{\Delta}^+$, therefore, by subsection (2.4.1), s is the unique element of minimal length that maps $\delta - \theta_\Sigma$ to $2\delta - \theta_\Sigma$. We study $N(s)$. For each $\beta \in \Delta^+(A(\Sigma))$,

$$s(\beta) = \beta + (\beta, \theta_\Sigma^\vee)(\delta - \theta_\Sigma)$$

hence $s(\beta) < 0$ if and only if $(\beta, \theta_\Sigma^\vee) < 0$. Thus if $(\beta, \theta_\Sigma^\vee) = -2$, then $\beta \in N(s)$. It remains to prove the converse. Assume $s(\beta) < 0$, hence $(\beta, \theta_\Sigma^\vee) < 0$: since $(\alpha, \theta_\Sigma) \geq 0$ for all $\alpha \in \widehat{\Pi} \setminus \Pi_1$, this implies that $ht_\sigma(\beta) \geq 1$. Now we observe that, if $\beta \in N(s)$, then also $-s(\beta) \in N(s)$, therefore $ht_\sigma(-s(\beta)) \geq 1$ as well. Since

$$ht_\sigma(s(\beta)) = ht_\sigma(\beta) + (\beta, \theta_\Sigma^\vee)ht_\sigma(\delta - \theta_\Sigma) = ht_\sigma(\beta) + (\beta, \theta_\Sigma^\vee),$$

we obtain that $-(\beta, \theta_\Sigma^\vee) = ht_\sigma(\beta) + ht_\sigma(-s(\beta)) \geq 2$. But $k\delta - \theta_\Sigma$ belongs to $\mathcal{M}_\sigma \subset \Pi_0^*$, therefore, by (3.1), we have $-(\beta, \theta_\Sigma^\vee) \leq \frac{2\|\beta\|}{\|\theta_\Sigma\|} \leq 2\sqrt{2}$, so we can conclude that $-(\beta, \theta_\Sigma^\vee) = 2$ and $ht_\sigma(\beta) = ht_\sigma(-s(\beta)) = 1$.

Now we assume $k = 1$. By Remark 4.2, then either $\mathfrak{g}$ is of type B_n, $\Pi_1 = \{\alpha_{n-1}\}$ and $\theta_\Sigma = \alpha_n$, or $\mathfrak{g}$ is of type C_n, $\Pi_1 = \{\alpha_0, \alpha_n\}$, $\Sigma = \{\alpha_1, \ldots, \alpha_{n-1}\}$. In

the first case a straightforward check shows that $s = s_{n-1}\cdots s_2 s_0 s_1 s_2 \cdots s_{n-1} = s_{\alpha_0+\alpha_2+\ldots+\alpha_{n-1}} s_{\alpha_1+\alpha_2+\ldots+\alpha_{n-1}}$ maps α_n to $\delta-\alpha_n$, α_{n-1} to $\alpha_{n-1}+2\alpha_n-\delta$, fixes α_i, $i=2,\ldots,n-2$ and switches α_0 and α_1. A positive root γ is orthogonal to α_n if and only if $c_{\alpha_{n-1}}(\gamma)=c_{\alpha_n}(\gamma)$. Therefore s keeps positive any positive root orthogonal to α_n, as required. It is clear that s is an involution, being conjugated to s_0s_1. A direct computation shows that $N(s)=\{\beta\in\Delta^+(\widehat{\Pi}\setminus\{\alpha_n\})\mid c_{\alpha_{n-1}}(\beta)=1\}=\{\beta\in\widehat{\Delta}^+_1\mid(\beta,\theta_\Sigma^\vee)=-2\}$.

For $\mathfrak{g}$ of type C_n, $s=s_0s_n$ maps $\theta_\Sigma=\alpha_1+\cdots+\alpha_{n-1}$ to $\delta-\theta_\Sigma=\alpha_0+\cdots+\alpha_n$. Moreover, a root in $\widehat{\Delta}^+$ is orthogonal to θ_Σ if and only if it is of the form $A\cup(\mathbb{N}\delta\pm A)$ where A is formed by the roots in the subsystem generated by $\alpha_2,\ldots,\alpha_{n-2}$ and by the roots $2\alpha_i+\ldots+2\alpha_{n-1}+\alpha_n, 2\le i\le n-1$ and $\alpha_1+\ldots+\alpha_n$. A direct check shows that these roots are kept positive by s, which is therefore minimal. It is immediate to see that $N(s)=\{\alpha_0,\alpha_n\}=\{\beta\in\widehat{\Delta}^+_1\mid(\beta,\theta_S^\vee)=-2\}$. □

Lemma 4.4. *Assume $\Sigma|\Pi_0$, $k\delta-\theta_\Sigma\in\mathcal{M}_\sigma$, $\alpha\in\widehat{\Pi}$, and $\|\alpha\|=\|\theta_\Sigma\|$.*

(1) *If θ_Σ is of type 1, $\alpha\in A(\Sigma)$, and w_α is the element of minimal length such that $w_\alpha(\alpha)=k\delta-\theta_\Sigma$, then $w_\alpha\in\mathcal{W}_\sigma^{ab}$.*

(2) *If θ_Σ is of type 2, $\alpha\in\Sigma$, v_α is the element of minimal length in $W(\Sigma)$ such that $v_\alpha(\alpha)=\theta_\Sigma$, and s is the element of minimal length in $\widehat{W}$ such that $s(\theta_\Sigma)=k\delta-\theta_\Sigma$, then $sv_\alpha\in\mathcal{W}_\sigma^{ab}$. Moreover, $\ell(sv_\alpha)=\ell(s)+\ell(v_\alpha)$ and sv_α is the element of minimal length in $\widehat{W}$ that maps α to $k\delta-\theta_\Sigma$.*

Proof. (1). By Proposition 4.2 (1) and Proposition 2.1 (5), if $\beta\in N(w_\alpha)$, then there exists $\beta'\in\widehat{\Delta}^+$ such that $\beta+\beta'=k\delta-\theta_\Sigma$. By Remark 3.1, each root less than μ in the usual root order has σ-height strictly less than 2, hence $ht_\sigma(\beta)=ht_\sigma(\beta')=1$.

(2). Assume first $k=2$, so that $s=s_{\delta-\theta_\Sigma}$. We first show that $s_{\delta-\theta_\Sigma}(\beta)=\beta+\delta-\theta_\Sigma$ for each $\beta\in N(v_\alpha)$. This amounts to prove that $(\theta_\Sigma^\vee,\beta)=1$ for each $\beta\in N(v_\alpha)$, which follows again from Proposition 2.1, (2). Thus we obtain that the σ-height of the roots in $s_{\delta-\theta_\Sigma}(N(v_\alpha))$ is 1; moreover,

$$N(sv_\alpha)=N(s_{\delta-\theta_\Sigma})\cup s_{\delta-\theta_\Sigma}(N(v_\alpha))$$

and $\ell(sv_\alpha)=\ell(s)+\ell(v_\alpha)$. Since by Lemma 4.3, for each $\beta\in N(s)$, $ht_\sigma(\beta)=1$, we conclude that $sv_\alpha\in\mathcal{W}_\sigma^{ab}$. It remains to prove the assertion about the minimal length. Notice that the above considerations show in particular that, for each $\beta\in N(sv_\alpha)$, we have that $(\beta,k\delta-\theta_\Sigma)\neq0$. By subsection 2.4.1, it follows that sv_α is the unique element of minimal length that maps α to $k\delta-\theta_\Sigma$.

In the case of B_n, one has $N(sv_\alpha)=N(s)=\{\beta\in\Delta^+(\widehat{\Pi}\setminus\{\alpha_n\})\mid c_{\alpha_{n-1}}(\beta)=1\}$. This follows noting that $L(s)=\{\alpha_{n-1}\},\ell(s)=2n-2=|\Delta^+(\widehat{\Pi}\setminus\{\alpha_n\})|-|\Delta^+(\widehat{\Pi}\setminus\{\alpha_{n-1},\alpha_n\})|$.

In the case C_n, we first remark that $sv_{\alpha_i}=s_0\cdots s_{i-1}s_n\cdots s_{i+1}$, $1\le i\le n-1$. Thus,

$$\begin{aligned}N(sv_{\alpha_i})&=N(s_0\cdots s_{i-1}s_n\cdots s_{i+1})\\&=\{\alpha_0+\ldots+\alpha_k\mid 0\le k\le i-1\}\cup\{\alpha_h+\ldots+\alpha_n\mid i+1\le h\le n\},\end{aligned}$$

whose elements have clearly σ-height 1. The same argument used in case $k=2$ proves that also in this case sv_α is the unique element of minimal length that maps α to $k\delta-\theta_\Sigma$. □

Lemma 4.5. *Assume $\mu \in \mathcal{M}_\sigma$, $\alpha \in \widehat{\Pi}$, and $w \in \mathcal{I}_{\alpha,\mu}$. Then*

(1) *for each $\beta \in N(w)$, $\mu + \beta \notin N(w)$;*
(2) *for each $\beta, \beta' \in \widehat{\Delta}^+$ such that $\beta + \beta' = \mu$, exactly one of β, β' belongs to $N(w)$.*

Proof. (1). We have

$$N(ws_\alpha) = N(w) \cup \{\mu\}. \tag{4.2}$$

If, for some $\beta \in N(w)$, $\beta + \mu \in \widehat{\Delta}^+$, then by the convexity properties, we would obtain $\beta + \mu \in N(w)$: this cannot happen since $ht_\sigma(\beta + \mu) \geq 3$, while w is σ-minuscule.

(2). By the convexity properties, relation (4.2) implies that $N(ws_\alpha)$ contains at least one summand of each decomposition $\mu = \beta + \beta'$, hence $N(w)$ does. Since $\mu \notin N(w)$, it contains exactly one summand. □

Lemma 4.6. *Assume $\mu = k\delta - \theta_\Sigma \in \mathcal{M}_\sigma$, $\alpha \in \widehat{\Pi}$, and $w \in \mathcal{I}_{\alpha,\mu}$. Then there exists $u \in \widehat{W}$ such that*

$$\{\beta \in N(w) \mid \mu - \beta \in \widehat{\Delta}^+\} = N(u).$$

In particular, $u \leq w$. Moreover, u belongs to $\mathcal{I}_{\alpha,\mu}$.

Proof. Set $U = \{\beta \in N(w) \mid \mu - \beta \in \widehat{\Delta}^+\}$. We first prove the existence of u: we have only to check that U is biconvex. We observe that, if $\beta, \beta' \in U$, then $\beta + \beta'$ is not a root, otherwise it would belong to $N(w)$, which impossible since $ht_\sigma(\beta + \beta') = 2$ and w is σ-minuscule. Thus we have only to check that, if $\beta \in U$ and $\beta = \gamma + \gamma'$, then at least one of γ, γ' belongs to U. Clearly, at least (in fact exactly) one of γ, γ', say γ, belongs to $N(w)$. We have to prove that $\mu - \gamma$ is a positive root. Set $\beta' = \mu - \beta$: by definition, β' is a positive root and it is immediate that $ht_\sigma(\beta') = 1$. Since $\gamma + \gamma' + \beta' = \mu$, at least one of $\gamma + \beta'$, $\gamma' + \beta'$, is a root, otherwise, by the Jacobi identity, $\gamma + \gamma' + \beta'$ would not be a root. But $\gamma + \beta'$ cannot be a root, otherwise it would have σ-height equal to 2, while being less than μ. Therefore $\mu - \gamma = \gamma' + \beta'$ is a root, as required.

It remains to prove that $u \in \mathcal{I}_{\alpha,\mu}$. It is clear that $u \in \mathcal{W}_\sigma^{ab}$, we have only to check that $u(\alpha) = \mu$. By Lemma 4.5 (2), $N(w)$ contains exactly one summand of any decomposition of μ as a sum of two positive roots and, by the definition of u, $N(u)$ has the same property. From this fact, we easily deduce that $N(u) \cup \{\mu\}$ is biconvex, hence that there exist a simple root $\beta \in \widehat{\Pi}$ such that $N(us_\beta) = N(u) \cup \{\mu\}$. But $N(us_\beta) = N(u) \cup \{u(\beta)\}$, hence $u(\beta) = \mu$. We must prove that $\beta = \alpha$. Since $u \leq w$, there exists $z \in \widehat{W}$ such that $w = uz$ and $N(w) = N(u) \cup uN(z)$. If $\beta \neq \alpha$, since $w(\beta) = uz(\beta) \neq \mu$, we obtain that $z(\beta) \neq \beta$, hence, by formula (2.3), that $N(z)$ contains at least one root γ such that $\gamma \not\perp \beta$. Then $u(\gamma) \not\perp \mu$ and $u(\gamma) \in N(w) \setminus N(u)$: we show that this is a contradiction. In fact, $u(\gamma) \not\perp \mu$ implies that either $\mu + u(\gamma)$ or $\mu - u(\gamma)$ is a positive root: the first instance is impossible by Lemma 4.5 (1); the second one is impossible because it would imply that $u(\gamma) \in N(u)$. □

Assume $\mu = k\delta - \theta_\Sigma$. In Lemma 4.4 we have constructed elements w_α and sw_α belonging to $\mathcal{I}_{\alpha,\mu}$, under certain restrictions on α. In particular, we have proved that, under such restrictions, $\mathcal{I}_{\alpha,\mu}$ is not empty. In the next proposition we prove

that if $\mathcal{I}_{\alpha,\mu}$ is not empty, then α must satisfy the conditions of Lemma 4.4 (1) (resp. (2)) and the element u built in Lemma 4.6 is actually w_α (resp. sv_α). We have therefore determined necessary and sufficient conditions under which $\mathcal{I}_{\alpha,\mu}$ is not empty.

Proposition 4.7. *Assume $\Sigma|\Pi_0$, $k\delta-\theta_\Sigma\in\mathcal{M}_\sigma$, $\alpha\in\widehat{\Pi}$, and $w\in\mathcal{I}_{\alpha,k\delta-\theta_\Sigma}$.*

(1) *If θ_Σ is of type 1, then $\alpha\in A(\Sigma)$ and $w\geq w_\alpha$.*
(2) *If θ_Σ is of type 2, then $\alpha\in\Sigma$ and $w\geq sv_\alpha$.*

Proof. (1). Set $\mu=k\delta-\theta_\Sigma$ and consider the element u built in Lemma 4.6. By Lemma 4.5 (2) and by the definition of u, $N(u)$ contains exactly one summand of any decomposition of μ as a sum of two positive roots, and each element of $N(u)$ is one of the summands of such a decomposition. By Proposition 4.2 and by Proposition 2.1 (6), there exists a simple root $\beta\in A(\Sigma)$ such that $u(\beta)=\mu$, and u is the minimal length element with this property. But $u(\alpha)=\mu$, hence $\alpha=\beta\in A(\Sigma)$, and $u=w_\alpha$.

(2). As above, we set $\mu=k\delta-\theta_\Sigma$ and consider the element u built in Lemma 4.6. We claim that in this case $\alpha\in\Sigma$ and $u=sv_\alpha$, which clearly implies the thesis.

We start proving that $s<u$, which, by Lemma 4.3, consists in proving that all $\beta\in\widehat{\Delta}_1^+$ such that $(\beta,\mu^\vee)=2$ belong to $N(u)$. Assume $\beta\in\widehat{\Delta}_1^+$ and $(\beta,\mu^\vee)=2$: this imply that $\mu-\beta$ and $2\mu-\beta$ are roots, and positive, having positive σ-height. By Lemma 4.5 (1), $\mu-\beta\notin N(u)$, since $\mu-\beta+\mu$ is a root, hence $\beta\in N(u)$. So $s<u$, i.e. there exists $v\in\widehat{W}$ such that $u=sv$ and $N(u)=N(s)\cup sN(v)$. It remains to prove that $\alpha\in\Sigma$ and $v=v_\alpha$.

First, we prove that for all $\beta\in N(u)$, we have that $(\beta,\mu^\vee)>0$. Assume by contradiction that $\beta\in N(u)$ and $(\beta,\mu^\vee)=0$, and set $\beta'=\mu-\beta$. Then $ht_\sigma(\beta')=1$ and $(\beta',\mu^\vee)=2$: by the previous part, this implies $\beta'\in N(u)$, which is impossible. Therefore we have $(\beta,\mu^\vee)>0$, hence $(\beta,\mu^\vee)\in\{1,2\}$, since $\mu\in\mathcal{M}_\sigma$. It follows that $sN(v)\subseteq\{\beta\in\widehat{\Delta}_1^+\mid(\beta,\mu^\vee)=1\}$.

Now, we claim that $N(v)\subseteq\Delta(\Sigma)$ and that, for each $\beta\in\Delta^+(\Sigma)$ such that $\theta_\Sigma-\beta$ is positive, exactly one among β and $\theta_\Sigma-\beta$ belongs to $N(v)$. Assume $\beta\in N(v)$ and set $\beta'=s(\beta)$. Then $ht_\sigma(\beta')=1$ and $(\beta',\mu^\vee)=1$, so that $\mu-\beta'$ is a positive root, $(\mu-\beta',\mu^\vee)=1$, and $ht_\sigma(\mu-\beta')=1$. By the explicit description of $N(s)$, $\theta_\Sigma-\beta=s(\mu-\beta')$ is positive, hence $\beta\in\Delta(\Sigma)$. Now let $\beta\in\Delta(\Sigma)^+$ be such that $\theta_\Sigma-\beta\in\Delta(\Sigma)^+$ and set $\beta'=s(\beta)$. Then, θ_Σ being long with respect to $\Delta(\Sigma)$, we obtain that $(\beta,\theta_\Sigma^\vee)=(\theta_\Sigma-\beta,\theta_\Sigma^\vee)=1$, hence $(\beta',\mu^\vee)=(\mu-\beta',\mu^\vee)=1$. Moreover, by the explict description of $N(s)$, both β' and $\mu-\beta'$ are positive, therefore both have σ-height equal to 1. By Lemma 4.5 (2), it follows that exactly one among them belongs to $N(u)$, hence to $sN(v)$, therefore, exactly one among β and $\theta_\Sigma-\beta$ belongs to $N(v)$. Thus v has the property of Proposition 2.1 (6), whence there exists $\beta\in\Sigma$ such that $v=v_\beta$. But $sv_\beta(\beta)=\mu$, hence $\beta=\alpha$. □

We have finally to deal with the posets $\mathcal{I}_{\alpha,\mu}$ with $\mu=\beta+k\delta$, $\beta\in\Pi_1$. According to our definitions, (3.2) and (3.3), the assumption that $\beta+k\delta\in\mathcal{M}_\sigma$ implies that β is long.

We start refining the analysis done in [3, Lemma 5.10].

Proposition 4.8. *If $\mathfrak{g}^{\bar{0}}$ is semisimple, then $\mathfrak{g}^{\bar{1}}$ is irreducible as a $\mathfrak{g}^{\bar{0}}$-module. If $\mathfrak{g}^{\bar{0}}$ is not semisimple, then $\mathfrak{g}^{\bar{1}}$ has two irreducible components as a $\mathfrak{g}^{\bar{0}}$-module. As a consequence, the following holds. Denote by w_0 the longest element of W_0. Then*

1. *if $\Pi_1 = \{\alpha\}$ then $w_0(\alpha) = k\delta - \alpha$;*
2. *if $\Pi_1 = \{\alpha, \beta\}$, then $w_0(\alpha) = \delta - \beta$ and $w_0(\beta) = \delta - \alpha$.*

Proof. It is well-known that $t^{-1} \otimes \mathfrak{g}^{\bar{1}}$ occurs as a submodule of the homology $H_1(\mathfrak{u}^-)$ where $\mathfrak{u}^- = \sum_{\alpha \in (-\widehat{\Delta}^+)\setminus\Delta_0} \widehat{L}(\mathfrak{g}, \sigma)_\alpha$. By Garland-Lepowsky theorem, this homology decomposes as $\oplus_{\alpha\in\Pi_1} V(-\alpha)$, as a sum of irreducible $(\mathfrak{g}^{\bar{0}} + \mathbb{C}K + \mathbb{C}d)$-modules, which stay irreducible as $\mathfrak{g}^{\bar{0}}$-modules. It follows that

$$t^{-1} \otimes \mathfrak{g}^{\bar{1}} = V(-\alpha), \text{ if } \Pi_1 = \{\alpha\}. \tag{4.3}$$

Moreover, it is clear that $-\alpha$ occurs as a highest weights of $t^{-1} \otimes \mathfrak{g}^{\bar{1}}$, for any $\alpha \in \Pi_1$, hence,

$$t^{-1} \otimes \mathfrak{g}^{\bar{1}} = V(-\alpha) \oplus V(-\beta), \text{ if } \Pi_1 = \{\alpha, \beta\} \text{ with } \alpha \neq \beta. \tag{4.4}$$

Since $\mathfrak{g}^{\bar{1}}$ is self-dual as a $\mathfrak{g}^{\bar{0}}$-module, if $\Pi_1 = \{\alpha\}$ we obtain that $w_0(\bar{\alpha}) = -\bar{\alpha}$, hence $w_0(\alpha) = w_0(\delta' + \bar{\alpha}) = \delta' - \bar{\alpha} = 2\delta' - \alpha = k\delta - \alpha$ (cf. Section 2.1), as claimed.

Assume $\Pi_1 = \{\alpha, \beta\}$. Notice that in this case $k = 1$, so that $\delta = 2\delta'$ and that $c_\alpha(\delta) = c_\beta(\delta) = 1$ (see Section 2.1). We have two cases:

1. $V(-\alpha)^* = V(-\alpha)$ and $V(-\beta)^* = V(-\beta)$,
2. $V(-\alpha)^* = V(-\beta)$ and $V(-\beta)^* = V(-\alpha)$.

In the first case we have $w_0(\bar{\alpha}) = -\bar{\alpha}$, which forces $w_0(\alpha) = w_0(\delta' + \bar{\alpha}) = \delta' - \bar{\alpha} = \delta - \alpha$ and this is not possible since $c_\alpha(w_0(\alpha)) = c_\alpha(\alpha) = 1$, while $c_\alpha(\delta - \alpha) = 0$. Hence (2) holds. It follows that $w_0(\bar{\alpha}) = -\bar{\beta}$ and $w_0(\bar{\beta}) = -\bar{\alpha}$. Therefore, $w_0(\alpha) = \delta' - \bar{\beta} = 2\delta' - \beta = \delta - \beta$ and $w_0(\beta) = \delta' - \bar{\alpha} = 2\delta' - \alpha = \delta - \alpha$. □

Proposition 4.9. *Assume $\mu = \alpha + k\delta \in \mathcal{M}_\sigma$, with $\alpha \in \Pi_1$. Set $\Pi_{0,\alpha} = \Pi_0 \cap \alpha^\perp$, $W_{0,\alpha} = W(\Pi_{0,\alpha})$, and denote by $w_{0,\alpha}$ the longest element of $W_{0,\alpha}$.*

1. *If $\Pi_1 = \{\alpha\}$, then $\mathcal{I}_{\gamma,\mu} \neq \emptyset$ if and only if $\gamma = \alpha$. Moreover,*
$$\mathcal{I}_{\alpha,\mu} = \{s_\alpha w_{0,\alpha} w_0\}.$$
2. *If $\Pi_1 = \{\alpha, \beta\}$, then $\mathcal{I}_{\gamma,\alpha+k\delta} \neq \emptyset$ if and only if $\gamma = \beta$. Moreover,*
$$\min \mathcal{I}_{\beta,\alpha+k\delta} = s_\alpha w_{0,\alpha} w_0.$$

Proof. Set $x = s_\alpha w_{0,\alpha} w_0$. By Proposition 4.8, we have that:
(1) if $\Pi_1 = \{\alpha\}$, then $x(\alpha) = s_\alpha w_{0,\alpha} w_0(\alpha) = s_\alpha w_{0,\alpha}(k\delta - \alpha) = s_\alpha(k\delta - \alpha) = k\delta + \alpha$;
(2) if $\Pi_1 = \{\alpha, \beta\}$, then $x(\beta) = s_\alpha w_{0,\alpha} w_0(\beta) = s_\alpha w_{0,\alpha}(k\delta - \alpha) = s_\alpha(k\delta - \alpha) = k\delta + \alpha$.

We prove that x is σ-minuscule. Since $w_{0,\alpha} w_0 \in W_0$, it is clear that $N(w_{0,\alpha} w_0) \subseteq \Delta_0^+$. In fact, we have $N(w_{0,\alpha} w_0) = \Delta_0^+ \setminus \Delta(\Pi_{0,\alpha})$. Since α is long, for each $\gamma \in \Delta_0^+ \setminus \Delta(\Pi_{0,\alpha})$, we have $s_\alpha(\gamma) = \gamma + \alpha$, hence $N(x) = \{\alpha\} \cup s_\alpha N(w_{0,\alpha} w_0) \subseteq \widehat{\Delta}_1$, as claimed.

So we have proved that $x \in \mathcal{I}_{\alpha,\alpha+k\delta}$, if $\Pi_1 = \{\alpha\}$, and $x \in \mathcal{I}_{\beta,\alpha+k\delta}$, if $\Pi_1 = \{\alpha, \beta\}$.

Now we treat separately the two cases. First, let $\Pi_1 = \{\alpha\}$ and assume that $w \in \mathcal{I}_{\gamma,\alpha+k\delta}$, with $\gamma \in \widehat{\Pi}$. Then $N(ws_\gamma) = N(w) \cup \{\alpha + k\delta\}$, hence, since w is σ-minuscule,

$$w(C_1) \subseteq \bigcap_{\eta\in\widehat{\Pi}_0} H_\eta^+ \setminus \bigcap_{\eta\in\Phi_\sigma} H_\eta^+ = P_\sigma \setminus D_\sigma,$$

where we denote by P_σ the polytope $\bigcap_{\eta\in\widehat{\Pi}_0} H_\eta^+$. But by [3, Lemma 5.11], there is exactly one $w\in\widehat{W}$ such that $w(C_1)\subseteq P_\sigma\setminus D_\sigma$, hence $w=x$, $\gamma=\alpha$, and $\mathcal{I}_{\alpha,\alpha+k\delta}=\{x\}$.

Now we assume $\Pi_1=\{\alpha,\beta\}$, $\gamma\in\widehat{\Pi}$ and $w\in\mathcal{I}_{\gamma,\delta+\alpha}$. We will show that $\gamma=\beta$ and $x\le w$. By Remark 3.4 (2) both roots in Π_1 are long; moreover, $\delta-\alpha$ is the highest root of $\Delta(\widehat{\Pi}\setminus\{\alpha\})$. For any $\gamma\in\widehat{\Pi}\setminus\{\alpha\}$, let v_γ be the element of minimal length that maps γ to $\delta-\alpha$. We start proving that $w_{0,\alpha}w_0=v_\beta$. In fact, it is clear that $w_{0,\alpha}w_0(\beta)=\delta-\alpha$, so it suffices then to check that $(w_{0,\alpha}w_0)^{-1}(\gamma)>0$ for all $\gamma\in\widehat{\Pi}$ such that $(\alpha,\gamma)=0$. If $\gamma\in\Pi_{0,\alpha}$ then $(w_{0,\alpha}w_0)^{-1}(\gamma)=w_0w_{0,\alpha}(\gamma)>0$. Moreover, in any case $(w_{0,\alpha}w_0)^{-1}(\beta)>0$, since $N(w_{0,\alpha}w_0)\subset\Delta_0$. Thus we obtain $w_{0,\alpha}w_0=v_\beta$, $x=s_\alpha v_\beta$, and $N(x)=\{\alpha\}\cup s_\alpha(N(v_\beta))$.

Now we consider w. Since $w(\gamma)=\delta+\alpha$, we have $w^{-1}(\alpha)=-\delta+\gamma$ hence $\alpha\in N(w)$. It follows that $w=s_\alpha z$ with $\ell(w)=1+\ell(z)$. In particular, $N(w)=\{\alpha\}\cup s_\alpha(N(z))$. Since $z(\gamma)=\delta-\alpha$, we have that $N(zs_\gamma)=N(z)\cup\{\delta-\alpha\}$, so the biconvexity of $N(zs_\gamma)$ implies that for any pair $\eta_1,\eta_2\in\widehat{\Delta}^+$ such that $\eta_1+\eta_2=\delta-\alpha$ exactly one of η_1,η_2 belong to $N(z)$. Moreover, $\delta-\alpha$ being a long root, for any such pair of roots we have $(\eta_i,(\delta-\alpha)^\vee)=1$, for $i=1,2$, since $(\eta_1+\eta_2,(\delta-\alpha)^\vee)=2$ and $(\eta_i,(\delta-\alpha)^\vee)\le 1$, for $i=1,2$. It follows that $s_\alpha(\eta_i)=\eta_i+\alpha$ and therefore, that $ht_\sigma(s_\alpha(\eta_i))=ht_\sigma(\eta_i)+1$, for $i=1,2$. Now, if $\eta_i\in N(z)$, $s_\alpha(\eta_i)\in N(w)$, and we obtain that $ht_\sigma(\eta_i)=0$. But $ht_\sigma(\delta-\alpha)=1$, so that one of the η_i has σ-height equal to 1 and the other has σ-height equal to 0. This implies that for any pair $\eta_1,\eta_2\in\widehat{\Delta}^+$ such that $\eta_1+\eta_2=\delta-\alpha$, $N(z)$ contains exactly the summand η_i having σ-height equal to 0. This must hold in particular when we take $w=x$ and so $z=v_\beta$. In this case we clearly obtain that $N(v_\beta)$ is exactly the set of the summands of σ-height equal to 0 of all the decomposition of $\delta-\alpha$ as a sum of two positive roots. So, for a general w, we obtain that $N(v_\beta)\subseteq N(z)$, whence $w_{\beta,\alpha+\delta}=s_\alpha v_\beta\le s_\alpha z=w$ as desired.

It remains to prove that $\gamma=\beta$. We have $z=v_\beta y$ with $N(z)=N(v_\beta)\cup v_\beta N(y)$, and $y(\gamma)=\beta$. If $\gamma\neq\beta$, then $N(y)$ would contain some roots not orthogonal to β, whence $v_\beta N(y)$ contains some root η not orthogonal to $\delta-\alpha$, hence to α. It follows that $s_\alpha(\eta)=\eta\pm\alpha\in N(w)$. But $\eta-\alpha\notin N(w)$, being summable to α that belongs to $N(w)$, hence $s_\alpha(\eta)=\eta+\alpha\in N(w)$. In particular, $ht_\sigma(\eta)=0$, and $\delta-\alpha-\eta\in\widehat{\Delta}^+$: this implies that $\eta\in N(v_\beta)$, a contradiction. $\square$

We sum up the results we have obtained in the following theorem.

If S is a connected subset of the set of simple roots, we denote by $S_{\overline{\ell}}$ the set of elements of S of the same length of θ_S. It is clear that, with respect to $\Delta(S)$, θ_S is a long root, therefore $S_{\overline{\ell}}$, is the set of the long roots of S, with respect to the subsystem $\Delta(S)$. With notation as in Lemma 4.4 and Proposition 4.9, we set

$$w_{\alpha,\mu}=\begin{cases} w_\alpha & \text{if } \mu=k\delta-\theta_\Sigma,\ \theta_\Sigma \text{ is of type 1, and } \alpha\in A(\Sigma)_{\overline{\ell}}\\ sv_\alpha & \text{if } \mu=k\delta-\theta_\Sigma,\ \theta_\Sigma \text{ is of type 2, and } \alpha\in \Sigma_{\overline{\ell}}\\ s_\beta w_{0,\beta}w_0 & \text{if } \mu=\beta+k\delta,\ \beta\in\Pi_1\end{cases}\tag{4.5}$$

and

$$\widehat{\Pi}_\mu = \begin{cases} A(\Sigma)_{\overline{\ell}} & \text{if } \mu = k\delta - \theta_\Sigma \text{ and } \theta_\Sigma \text{ is of type 1} \\ \Sigma_{\overline{\ell}} & \text{if } \mu = k\delta - \theta_\Sigma \text{ and } \theta_\Sigma \text{ is of type 2} \\ \Pi_1 & \text{if } \mu = \beta + k\delta \text{ and } \{\beta\} = \Pi_1 \\ \Pi_1 \setminus \{\beta\} & \text{if } \mu = \beta + k\delta,\ \beta \in \Pi_1, \text{ and } |\Pi_1| = 2 \end{cases} \tag{4.6}$$

Theorem 4.10. *Assume $\mu \in \mathcal{M}_\sigma$ and $\alpha \in \widehat{\Pi}$. Then $\mathcal{I}_{\alpha,\mu} \neq \emptyset$ if and only if $\alpha \in \widehat{\Pi}_\mu$. Moreover,*

$$w_{\alpha,\mu} = \min \mathcal{I}_{\alpha,\mu}.$$

Proof. The claim follows directly from Lemma 4.4, Proposition 4.7, and Proposition 4.9. □

5. The poset structure of $\mathcal{I}_{\alpha,\mu}$

We now study the poset structure of the sets $\mathcal{I}_{\alpha,\mu}$. This study is motivated by the following result, that shows that the maximal elements of the sets $\mathcal{I}_{\alpha,\mu}$ are maximal in the whole poset $\mathcal{W}_\sigma^{ab}$ except when $\alpha \in \Pi_1$ and $\mu = k\delta - \theta_\Sigma$, $\Sigma|\Pi_0$.

Proposition 5.1. *Suppose $w \in \mathcal{I}_{\alpha,\mu}$ and $v \geq w$ with $v \in \mathcal{W}_\sigma^{ab}$. If $v \notin \mathcal{I}_{\alpha,\mu}$, then $\alpha \in \Pi_1$. In that case, write explicitly $\Pi_1 = \{\alpha, \beta\}$ (with $\beta = \alpha$ if $|\Pi_1| = 1$). Then $v \in \mathcal{I}_{\alpha,k\delta+\beta}$.*

Proof. If $v \notin \mathcal{I}_{\alpha,\mu}$, write $v = wxs_\gamma y$ with $wx \in \mathcal{I}_{\alpha,\mu}$, $wxs_\gamma \notin \mathcal{I}_{\alpha,\mu}$ and $\ell(v) = \ell(w) + \ell(x) + \ell(y) + 1$. Then $(\gamma, \alpha) < 0$. Set $(\alpha, \gamma^\vee) = -r$ and consider $wxs_\gamma s_\alpha$. We have

$$N(wxs_\gamma s_\alpha) = N(wxs_\gamma) \cup \{wx(\alpha + r\gamma)\} = N(wxs_\gamma) \cup \{\mu + rwx(\gamma)\}.$$

Note that $ht_\sigma(\mu + rwx(\gamma)) = ht_\sigma(\mu) + r$. Since the latter root is not simple, there exists $\eta \in \widehat{\Pi}$ such that $\mu + wx(\gamma) - \eta \in \widehat{\Delta}^+$. Since $N(wxs_\gamma) \subset \widehat{\Delta}_1$ and $N(wxs_\gamma s_\alpha)$ is convex, we have that $\eta \notin \Pi_0$. Hence $\mu + rwx(\gamma)$ is minimal in $\widehat{\Delta}_{ht_\sigma(\mu)+r}$. Now we use Remark 3.1 about minimal roots. If $ht_\sigma(\mu) + r = 2s$ with $s > 1$ then $\mu + rwx(\gamma) = ks\delta - \theta_\Sigma$ for some $\Sigma|\Pi_0$. But then, by convexity, $k\delta - \theta_\Sigma \in N(wxs_\gamma)$ which is absurd. If $ht_\sigma(\mu) + r = 2s + 1$ with $s > 1$ then $\mu + rwx(\gamma) = ks\delta + \beta$ for some $\beta \in \Pi_1$. But then, by convexity, $k\delta + \beta \in N(wxs_\gamma)$ which is absurd. Therefore $ht_\sigma(\mu) = 2$ and $r = 1$. It follows that there exists $\beta \in \Pi_1$ such that $\mu + wx(\gamma) = \beta + k\delta$. In turn, we deduce that $wxs_\gamma \in \mathcal{I}_{\alpha,k\delta+\beta}$. By Proposition 4.9, (1), we have $\alpha \in \Pi_1$ as claimed, and $wxs_\gamma \in \mathcal{I}_{\alpha,k\delta+\beta}$ with $\beta = \alpha$ if $|\Pi_1| = 1$ and $\Pi_1 = \{\alpha, \beta\}$ otherwise. Since $v \geq wxs_\gamma \in \mathcal{I}_{\alpha,k\delta+\beta}$ and $ht_\sigma(k\delta + \beta) = 3$, by the first part of the proof, we have that $v \in \mathcal{I}_{\alpha,k\delta+\beta}$, as claimed. □

We now turn to the description of the poset structure of $\mathcal{I}_{\alpha,\mu}$: we will show that it is isomorphic to a poset $G'\backslash G$ for suitable reflection subgroups G, G' of $\widehat{W}$.

Definition 5.1. *For $\alpha \in \widehat{\Pi}$, and $\Sigma|\Pi_0$, we set*

$$\widehat{\Pi}_\alpha = \widehat{\Pi} \cap \alpha^\perp, \quad \widehat{W}_\alpha = W(\widehat{\Pi}_\alpha).$$

Lemma 5.2. *Let $\mu \in \mathcal{M}_\sigma$, $u, v \in \mathcal{I}_{\alpha,\mu}$, and $u < v$. Then $v = ux$ with $x \in \widehat{W}_\alpha$. In particular,*

$$\mathcal{I}_{\alpha,\mu} \subseteq w_{\alpha,\mu}\widehat{W}_\alpha.$$

Proof. By assumption, there exists $x \in \widehat{W}$ such that $N(v) = N(u) \cup uN(x)$: suppose by contradiction that $x \notin \widehat{W}_\alpha$. Then we may assume $x = x_1 s_\beta x_2$ with $\ell(x) = \ell(x_1) + \ell(x_2) + 1$, $x_1 \in \widehat{W}_\alpha$, and $\beta \in \widehat{\Pi}$, $\beta \not\perp \alpha$. Then $N(ux_1) \cup ux_1(\beta) \subseteq N(v)$. But $(\beta, \alpha) < 0$, hence $(ux_1(\beta), ux_1(\alpha)) = (ux_1(\beta), \mu) < 0$, therefore $ux_1(\beta) + \mu$ is a root: this cannot happen by Lemma 4.5 (1). □

By Lemma 5.2, $\mathcal{I}_{\alpha,\mu}$ is in bijection, in a natural way, with a subset of $\widehat{W}_\alpha$, namely, the subset of all $u \in \widehat{W}_\alpha$ such that $w_{\alpha,\mu} u \in \mathcal{I}_{\alpha,\mu}$. We will show that this subset is a system of minimal coset representatives of $\widehat{W}_\alpha$ modulo a certain subgroup $\widehat{W}_{\alpha,\mu}$. This will take the rest of the Section.

We start with giving a combinatorial characterization of the elements u such that $w_{\alpha,\mu} u \in \mathcal{I}_{\alpha,\mu}$.

Definition 5.2. *We set*

$$B_\mu = \begin{cases} \{\gamma \in \widehat{\Pi} \mid (\gamma, \theta_\Sigma^\vee) = 1\} & \text{if } \mu = k\delta - \theta_\Sigma \text{ and } \theta_\Sigma \text{ is of type 1,} \\ \Pi_1 & \text{if } \mu = k\delta - \theta_\Sigma \text{ and } \theta_\Sigma \text{ is of type 2,} \\ \{\beta\} & \text{if } \mu = k\delta + \beta,\ \beta \in \Pi_1, \end{cases}$$

$$V_{\alpha,\mu} = \{w \in \widehat{W}_\alpha \mid ht_{B_\mu}(\gamma) = 1\ \forall \gamma \in N(w)\},$$

if $B_\mu \neq \emptyset$. If $B_\mu = \emptyset$, we set $V_{\alpha,\mu} = \{1\}$.

Lemma 5.3. *Assume $\Sigma | \Pi_0$, $\mu = k\delta - \theta_\Sigma \in \mathcal{M}_\sigma$, and set*

$$B_\Sigma = \{\gamma \in \widehat{\Pi} \mid (\gamma, \theta_\Sigma) > 0\}.$$

Then

(1) *For all $\eta \in \widehat{\Delta}$,*

$$(\eta, \mu^\vee) = ht_\sigma(\eta) r_\Sigma - ht_{B_\Sigma}(\eta) \varepsilon_\Sigma, \tag{5.1}$$

where $\varepsilon_\Sigma = 2$, if $|\Sigma| = 1$, $\varepsilon_\Sigma = 1$, otherwise.

(2) *$B_\Sigma = \{\gamma \in \widehat{\Pi} \mid (\gamma, \theta_\Sigma^\vee) = 1\}$, unless $|\Sigma| = 1$.*

Proof. It is clear that for $\gamma \in \widehat{\Pi}$, we have $(\gamma, \theta_\Sigma) < 0$ if and only if $\gamma \in \Pi_1$; moreover, recall that $r_\Sigma = -(\gamma, \theta_\Sigma^\vee) = (\gamma, \mu^\vee)$ for all $\gamma \in \Pi_1$.

On the other hand, by definition, we have $(\gamma, \theta_\Sigma) > 0$ if and only if $\gamma \in B_\Sigma$. Clearly, $B_\Sigma \subseteq \Sigma$, and since θ_Σ is long with respect to $\Delta(\Sigma)$, it follows that, if $\gamma \in B_\Sigma$, then $(\gamma, \theta_\Sigma^\vee) = 1$ unless $\Sigma = B_\Sigma = \{\theta_\Sigma\}$, in which case $(\gamma, \theta_\Sigma^\vee) = 2$. Therefore, for any $\eta \in \widehat{\Delta}$,

$$(\eta, \mu^\vee) = \sum_{\gamma \in \Pi_1} c_\gamma(\eta)(\gamma, \mu^\vee) + \sum_{\gamma \in B_\Sigma} c_\gamma(\eta)(\gamma, \mu^\vee) = ht_\sigma(\eta) r_\Sigma - ht_{B_\Sigma}(\eta) \varepsilon_\Sigma$$

as wished. □

Lemma 5.4. *Assume $\mathcal{I}_{\alpha,\mu} \neq \emptyset$. For any $u \in \widehat{W}$, $w_{\alpha,\mu} u \in \mathcal{I}_{\alpha,\mu}$ if and only if $u \in V_{\alpha,\mu}$.*

Proof. We deal with the three cases that occur in the definition of B_μ one by one. We shall use several times relation (5.1) from Lemma 5.3.

1. $\mu = k\delta - \theta_\Sigma$, θ_Σ of type 1. Then $\alpha \in A(\Sigma)$, μ is the highest root of $A(\Sigma)$, and $w_{\alpha,\mu} \in W(A(\Sigma))$. It is clear that $B_\Sigma \cap A(\Sigma) = \emptyset$, in fact, by Definition 4.1, $A(\Sigma)$ is a connected component of $\widehat{\Pi} \setminus B_\Sigma$. In particular, for all $\eta \in \widehat{\Delta}$, $ht_{B_\Sigma}(w_{\alpha,\mu}(\eta)) = ht_{B_\Sigma}(\eta)$. Recall that r_Σ is the type of θ_Σ. By (5.1),

$$\begin{aligned}(w_{\alpha,\mu}(\eta), \mu^\vee) &= ht_\sigma(w_{\alpha,\mu}(\beta)) - ht_{B_\Sigma}(w_{\alpha,\mu}(\eta))\varepsilon_\Sigma = \\ &= ht_\sigma(w_{\alpha,\mu}(\beta)) - ht_{B_\Sigma}(\eta)\varepsilon_\Sigma.\end{aligned}$$

Now, assume $u \in V_{\alpha,\mu}$. If $u = 1$, obviously $w_{\alpha,\mu}u \in \mathcal{I}_{\alpha,\mu}$. So we may assume $u \neq \mathbf{1}$ and $|\Sigma| > 1$. If $\eta \in N(u)$, then $(\eta, \alpha) = 0$, so that $(w_{\alpha,\mu}(\eta), \mu) = 0$; moreover, $ht_{B_\Sigma}(\eta) = \varepsilon_\Sigma = 1$. Therefore, by the above identities we obtain that $ht_\sigma(w_{\alpha,\mu}(\eta)) = 1$. Thus $N(w_{\alpha,\mu}u) = N(w_{\alpha,\mu}) \cup w_{\alpha,\mu}N(u) \subseteq \widehat{\Delta}_1$, hence $w_{\alpha,\mu}u \in \mathcal{W}_\sigma^{ab}$. Since $u(\alpha) = \alpha$, we conclude that $w_{\alpha,\mu}u \in \mathcal{I}_{\alpha,\mu}$.

Conversely, if $w_{\alpha,\mu}u \in \mathcal{I}_{\alpha,\mu}$ with $u \neq 1$, then, by Lemma 5.2, $u \in \widehat{W}_\alpha$, so that, if $\eta \in N(u)$, then $(\eta, \alpha) = 0$, hence $(w_{\alpha,\mu}(\eta), \mu) = 0$. Moreover, $ht_\sigma(w_{\alpha,\mu}(\eta)) = 1$. It follows that $\varepsilon_\Sigma = 1$ and $ht_{B_\Sigma}(\eta) = 1$, so $ht_{B_\Sigma}(\eta) = ht_{B_\mu}(\eta) = 1$, hence $u \in V_{\alpha,\mu}$.

2. $\mu = k\delta - \theta_\Sigma$, θ_Σ of type 2. Then $\alpha \in \Sigma$, and $w_{\alpha,\mu} = sv_\alpha$, where v_α is the minimal element that maps α to θ_Σ and s is the minimal element that maps θ_Σ to μ. We also know that s is an involution. In this case, $B_\mu = \Pi_1$, hence $B_\mu \cap \Sigma = \emptyset$. Thus the B_μ-height is the σ-height and, since $v_\alpha \in W(\Sigma)$, we have that v_α preserves the σ-height. Similarly, since $s \in W(A(\Sigma))$, s preserves the B_Σ-height. Therefore, for all $\eta \in \widehat{\Delta}$, we obtain that

$$\begin{aligned}(w_{\alpha,\mu}(\eta), \mu^\vee) &= (v_\alpha(\eta), s\mu^\vee) = (v_\alpha(\eta), \theta_\Sigma^\vee) = -(v_\alpha(\eta), \mu^\vee) \\ &= -2ht_\sigma(v_\alpha(\eta)) + ht_{B_\Sigma}(v_\alpha(\eta))\varepsilon_\Sigma \\ &= -2ht_\sigma(\eta) + ht_{B_\Sigma}(v_\alpha(\eta))\varepsilon_\Sigma,\end{aligned}$$

and also that

$$\begin{aligned}(w_{\alpha,\mu}(\eta), \mu^\vee) &= 2ht_\sigma(w_{\alpha,\mu}(\eta)) - ht_{B_\Sigma}(w_{\alpha,\mu}(\eta))\varepsilon_\Sigma \\ &= 2ht_\sigma(w_{\alpha,\mu}(\eta)) - ht_{B_\Sigma}(v_\alpha(\eta))\varepsilon_\Sigma.\end{aligned}$$

In particular, if $(\mu^\vee, w_{\alpha,\mu}(\eta)) = 0$, then $ht_\sigma(w_{\alpha,\mu}(\eta)) = ht_\sigma(\eta) = ht_{B_\mu}(\eta)$. By Lemma 5.2, this directly implies that $w_{\alpha,\mu}u \in \mathcal{I}_{\alpha,\mu}$ if and only if $u \in V_{\alpha,\mu}$.

3. $\mu = k\delta + \beta$, $\beta \in \Pi_1$. If $|\Pi_1| = 1$, then $V_{\alpha,\mu} = \{1\}$ and, by Proposition 4.9, $\mathcal{I}_{\alpha,\mu} = \{w_{\alpha,\mu}\}$. So we may assume $|\Pi_1| = 2$, $\Pi_1 = \{\alpha, \beta\}$. Then, with notation as in Proposition 4.9, we have that $w_{\alpha,\mu} = s_\beta v_\beta$. Since $v_\beta(\alpha) = \delta - \beta$, we deduce that $v_\beta^{-1}(\beta) = \delta - \alpha$, hence, if $(\gamma, \alpha) = 0$, then

$$s_\beta v_\beta(\gamma) = v_\beta(\gamma) - (v_\beta(\gamma), \beta^\vee)\beta = v_\beta(\gamma) - (\gamma, \delta - \alpha^\vee)\beta = v_\beta(\gamma).$$

It follows that, if $\gamma \in \widehat{\Delta}_\alpha^+$, then $ht_\sigma(w_{\alpha,\mu}(\gamma)) = ht_\sigma(v_\beta(\gamma)) = c_\beta(\gamma) = ht_{B_\mu}(\gamma)$ and we can argue as in case 2. $\square$

Lemma 5.5. *Assume $\alpha \in \widehat{\Pi}$, $\mu \in \mathcal{M}_\sigma$, $\mathcal{I}_{\alpha,\mu} \neq \emptyset$, $B_\mu \neq \emptyset$, and set*

$$\Delta^2_{\alpha,\mu} = \{\gamma \in \Delta(\widehat{\Pi}_\alpha) \mid ht_{B_\mu}(\gamma) \geq 2\}.$$

Then $\Delta^2_{\alpha,\mu} \neq \emptyset$ if and only if $\mu = k\delta - \theta_\Sigma$ with θ_Σ of type 1 and $|\Sigma| > 1$, and $\alpha \in A(\Sigma) \setminus (\Sigma \cup \Pi_1)$. In this case, θ_Σ is the minimal element in $\Delta^2_{\alpha,\mu}$, with respect to the usual root order. Moreover, $ht_{B_\mu}(\theta_\Sigma) = ht_{B_\Sigma}(\theta_\Sigma) = 2$.

Proof. We deal with the three cases that occur in the definition of B_μ one by one.

1. $\mu = k\delta - \theta_\Sigma$, θ_Σ of type 1. Then $\alpha \in A(\Sigma)$ and $|\Sigma| > 1$, since we are assuming $B_\mu \neq \emptyset$. In particular $B_\Sigma = B_\mu = \{\gamma \in \widehat{\Pi} \mid (\gamma, \theta_\Sigma^\vee) = 1\}$.

We first prove that if $\gamma \in \Delta(\widehat{\Pi}_\alpha)$ and $ht_{B_\mu}(\gamma) \geq 2$ then $\gamma \geq \theta_\Sigma$. We notice $(\beta, \theta_\Sigma^\vee) \in \{0, 1\}$ for any $\beta \in \Delta^+(\Sigma) \setminus \{\theta_\Sigma\}$, therefore, since $(\theta_\Sigma, \theta_\Sigma^\vee) = 2$, $ht_{B_\mu}(\theta_\Sigma) = 2$ and θ_Σ is the unique root in $\Delta(\Sigma)$ with this property. It follows that we can assume $\gamma \notin \Delta(\Sigma)$, so that $ht_\sigma(\gamma) > 0$. Since $c_\alpha(k\delta - \gamma) > 0$, we have that $k\delta - \gamma$ is a positive root, hence $ht_\sigma(\gamma) \leq 2$. If $ht_\sigma(\gamma) = 1$, then $(\gamma, \theta_\Sigma^\vee) = 1$, hence $\gamma - \theta_\Sigma$ is a root, which can't be negative, since γ is supported also outside Σ. So it is positive, hence $\gamma \geq \theta_\Sigma$. Suppose now $ht_\sigma(\gamma) = 2$. Then $k\delta - \gamma \in \Delta_0$, hence it should belong to the component Σ' of Π_0 to which α belongs, since $c_\alpha(k\delta - \gamma) > 0$. Thus $\gamma = k\delta - \beta$ with $\beta \in \Sigma'$. If $\Sigma = \Sigma'$, then $\alpha \in \Gamma(\Sigma)$. Let Z be the component of $\Gamma(\Sigma)$ containing α. Let $\eta \in \Pi_1$ be a root such that $(\eta, \theta_Z) < 0$. We have that $\eta + \theta_Z + \theta_\Sigma$ is a root, so $k\delta - \eta - \theta_\Sigma - \theta_Z$ is a root, which is positive since its σ-height is 1. It follows that $k\delta \geq \theta_\Sigma + \theta_Z$, hence $\gamma \geq \theta_\Sigma - \beta + \theta_Z$. But then $c_\alpha(\gamma) > 0$, which is impossible. We have therefore $\Sigma' \neq \Sigma$. But then $\gamma = k\delta - \beta$ with $\beta \notin \Sigma$, so, clearly, $\gamma \geq \theta_\Sigma$.

It remains only to check that $\Delta^2_{\alpha,\mu} \neq \emptyset$ if and only if $\alpha \in A(\Sigma) \setminus (\Sigma \cup \Pi_1)$. If $\alpha \in A(\Sigma) \setminus (\Sigma \cup \Pi_1)$ then $\theta_\Sigma \in \Delta(\widehat{\Pi}_\alpha)$, hence $\theta_\Sigma \in \Delta^2_{\alpha,\mu}$. Assume now $\gamma \in \Delta^2_{\alpha,\mu}$. If $\alpha \in \Pi_1 \cup \Sigma$, then $\theta_\Sigma \notin \Delta(\widehat{\Pi}_\alpha)$, hence $\gamma > \theta_\Sigma$. This is absurd since it implies $c_\alpha(\gamma) > 0$.

2. $\mu = k\delta - \theta_\Sigma$, θ_Σ of type 2. Then $\alpha \in \Sigma$ and $B_\mu = \Pi_1$, so that the B_μ-height is the σ-height. We shall prove that, if $\gamma \in \Delta(\widehat{\Pi}_\alpha)$, then $ht_\sigma(\gamma) \leq 1$.

Consider first the case $k = 2$. Assume $\gamma \in \Delta(\widehat{\Pi}_\alpha)$. Notice that, if $\delta - \gamma$ is a root, then it is positive, since then $c_\alpha(\delta - \gamma) > 0$. Since $ht_\sigma(\delta) = 1$, this implies that $ht_\sigma(\gamma) \leq 1$. Now, assume by contradiction that $ht_\sigma(\gamma) > 1$. Since, in any case, $2\delta - \gamma \in \widehat{\Delta}^+$, we obtain that $ht_\sigma(\gamma) = 2$ and $2\delta - \gamma \in \Delta_0^+$. In turn, this implies that $2\delta - \gamma \in \Delta(\Sigma)$, since $c_\alpha(2\delta - \gamma) > 0$, and $\alpha \in \Sigma$. Thus, since θ_Σ is of type 2, also $2\delta - \gamma$ is of type 2. But this implies that $\delta - \gamma$ is a root, hence that $ht_\sigma(\gamma) \leq 1$: a contradiction.

Next, consider the case $k = 1$. In case B_n, we have $\Sigma = \{\alpha_n\}$ and $\Pi_1 = \{\alpha_{n-1}\}$, so $\alpha = \alpha_n$ and and $ht_\sigma(\gamma) = 0$ for all $\gamma \in \Delta(\widehat{\Pi}_\alpha)$. In case C_n, we have $\Sigma = \{\alpha_1, \ldots, \alpha_{n-1}\}$ and $\Pi_1 = \{\alpha_0, \alpha_n\}$, so it is clear that for all $\alpha \in \Sigma$, and for all $\gamma \in \Delta(\widehat{\Pi}_\alpha)$, $ht_\sigma(\gamma) \leq 1$.

3. $\mu = k\delta + \beta$, $\beta \in \Pi_1$. In this case $\alpha \in \Pi_1$ and $B_\mu \subseteq \Pi_1$, so it is clear that, if $\Pi_1 = \{\alpha\}$, then $ht_\sigma(\gamma) = 0$ for all $\gamma \in \Delta(\widehat{\Pi}_\alpha)$. If $\Pi_1 = \{\alpha, \beta\}$, we obtain in any case that $ht_\sigma(\gamma) \leq 1$ for all $\gamma \in \Delta(\widehat{\Pi}_\alpha)$. □

Definition 5.3. *Given $\alpha \in \widehat{\Pi}$ and $\mu \in \mathcal{M}_\sigma$ such that $\mathcal{I}_{\alpha,\mu} \neq \emptyset$, we set*

$$\widehat{\Pi}_{\alpha,\mu} = \widehat{\Pi}_\alpha \setminus B_\mu,$$

$$\widehat{\Pi}^*_{\alpha,\mu}=\begin{cases}\widehat{\Pi}_{\alpha,\mu}\cup\{\theta_\Sigma\} & \text{if } \mu=k\delta-\theta_\Sigma, \theta_\Sigma \text{ of type 1, } |\Sigma|>1,\\ & \alpha\in A(\Sigma)\setminus(\Sigma\cup\Pi_1),\\ \widehat{\Pi}_{\alpha,\mu} & \text{in all other cases;}\end{cases}$$

$$\widehat{W}_{\alpha,\mu}=W(\widehat{\Pi}^*_{\alpha,\mu}).$$

The main results of this Section is the following statement. Recall that we identify a coset space with the set of minimal length coset representatives.

Theorem 5.6. *Let $\alpha\in\widehat{\Pi}$ and $\mu\in\mathcal{M}_\sigma$ be such that $\mathcal{I}_{\alpha,\mu}\neq\emptyset$. Then the map $u\mapsto w_{\alpha,\mu}u$ is a poset isomorphism between $\mathcal{I}_{\alpha,\mu}$ and $\widehat{W}_{\alpha,\mu}\backslash\widehat{W}_\alpha$.*

Proof. By Lemma 5.4, we have only to prove that $\widehat{W}_{\alpha,\mu}\backslash\widehat{W}_\alpha=V_{\alpha,\mu}$.

Let $u\in V_{\alpha,\mu}$, $u\neq\mathbf{1}$. To prove that $u\in\widehat{W}_{\alpha,\mu}\backslash\widehat{W}_\alpha$ we have to show that if $\beta\in\widehat{\Pi}^*_{\alpha,\mu}$, then $u^{-1}(\beta)\in\widehat{\Delta}^+$: this is immediate from the definitions, since $ht_{B_\mu}(\beta)\in\{0,2\}$, while, for all $\gamma\in N(u)$, $ht_{B_\mu}(\gamma)=1$.

Conversely, assume $u\in\widehat{W}_{\alpha,\mu}\backslash\widehat{W}_\alpha$, $u\neq\mathbf{1}$, and $\gamma\in N(u)$. If, by contradiction, $ht_{B_\mu}(\gamma)=0$, then, by the biconvexity property of $N(u)$, we obtain that there exists some $\beta\in(\widehat{\Pi}_\alpha\setminus B_\mu)\cap N(u)$: this contradicts the definition of $\widehat{W}_{\alpha,\mu}\backslash\widehat{W}_\alpha$. Therefore, $ht_{B_\mu}(\gamma)>0$. By Lemma 5.5, this implies that $ht_{B_\mu}(\gamma)=1$ in all cases except when $\mu=k\delta-\theta_\Sigma$, with θ_Σ of type 1 and $|\Sigma|>1$. It remains to prove that also in this case $ht_{B_\mu}(\gamma)=1$. First, we observe that, by Lemma 4.1, $ht_{B_\Sigma}(k\delta-\theta_\Sigma)=0$: it follows that $ht_{B_\Sigma}(k\delta)=ht_{B_\Sigma}(\theta_\Sigma)$ and, by Lemma 5.5, that $ht_{B_\mu}(k\delta)=2$. Hence, $ht_{B_\mu}(\gamma)\leq 2$, since $k\delta-\gamma$ is a positive root. Now, if we assume, by contradiction, that $ht_{B_\mu}(\gamma)=2$, then by Lemma 5.5, we obtain that γ is equal to θ_Σ plus a, possibly empty, sum of positive roots with null B_μ-height. By the biconvexity of $N(u)$, this implies that some root in $(\widehat{\Pi}_\alpha\setminus B_\mu)\cup\{\theta_\Sigma\}$ belongs to $N(u)$, in contradiction with the definition of $\widehat{W}_{\alpha,\mu}\backslash\widehat{W}_\alpha$. □

6. Intersections among $\mathcal{I}_{\alpha,\mu}$'s

Our goal in this Section is the proof of the following Theorem.

Theorem 6.1. *(1). If $\alpha\neq\beta$, then $\mathcal{I}_{\alpha,\mu}\cap\mathcal{I}_{\beta,\mu'}\neq\emptyset$ if and only if $\mu=k\delta-\theta_\Sigma$, $\mu'=k\delta-\theta'_\Sigma$ with $\Sigma\neq\Sigma'$, $\alpha\in\Sigma'$, $\beta\in\Sigma$, and $\alpha,\beta,\theta_\Sigma,\theta_{\Sigma'}$ all of type 1.*
(2). If $\mathcal{I}_{\alpha,\mu}\cap\mathcal{I}_{\beta,\mu'}\neq\emptyset$, then $\sup\{\min\mathcal{I}_{\alpha,\mu},\min\mathcal{I}_{\beta,\mu'}\}$ exists. Denoting it by $w_{\alpha,\beta}$, we have that

$$\mathcal{I}_{\alpha,\mu}\cap\mathcal{I}_{\beta,\mu'}\cong W((\widehat{\Pi}_\alpha\cap\widehat{\Pi}_\beta)\setminus\Pi_1)\backslash W(\widehat{\Pi}_\alpha\cap\widehat{\Pi}_\beta),$$

the isomorphism being $u\mapsto w_{\alpha,\beta}u$, $u\in W((\widehat{\Pi}_\alpha\cap\widehat{\Pi}_\beta)\setminus\Pi_1)\backslash W(\widehat{\Pi}_\alpha\cap\widehat{\Pi}_\beta)$.

Statements (1), (2) are proved in Propositions 6.7, 6.6, respectively.

Definition 6.1. *Assume that Σ and Σ' are distinct components of Π_0. We define*

$$A(\Sigma,\Sigma')=A(\Sigma)\cap A(\Sigma').$$

Moreover, we set

$$W_{\Sigma,\Sigma'}=W(A(\Sigma,\Sigma')),\qquad W^1_{\Sigma,\Sigma'}=W(A(\Sigma,\Sigma')\setminus\Pi_1),$$

and denote by $u_{\Sigma,\Sigma'}$ the maximal element in $W^1_{\Sigma,\Sigma'}\backslash W_{\Sigma,\Sigma'}$.

According to Definition 6.1 and Subsection 2.4.2,

$$N(u_{\Sigma,\Sigma'}) = \{\beta \in \Delta(A(\Sigma,\Sigma')) \mid ht_\sigma(\beta) > 0\}. \tag{6.1}$$

It is clear from Definition 4.1 that $\Sigma' \subseteq A(\Sigma)$; in fact, we have the partition

$$A(\Sigma) = \bigcup_{\substack{\Sigma'|\Pi_0 \\ \Sigma'\neq\Sigma}} \Sigma' \cup \Pi_1 \cup \Gamma(\Sigma). \tag{6.2}$$

From this, we obtain

$$A(\Sigma,\Sigma') = \Gamma(\Sigma) \cup \Pi_1 \cup \Gamma(\Sigma') \cup \Sigma'', \tag{6.3}$$

where $\Sigma'' = \Pi_0 \setminus (\Sigma \cup \Sigma')$. In particular we obtain the partition

$$A(\Sigma) = A(\Sigma,\Sigma') \cup (\Sigma' \setminus \Gamma(\Sigma')). \tag{6.4}$$

Remark 6.1. From equation (6.3) and Definition 4.1, we obtain directly that θ_Σ and $\theta_{\Sigma'}$ are orthogonal to all the roots in $A(\Sigma,\Sigma')$, except the ones in Π_1. This implies that $(\beta,\theta_\Sigma) \le 0$ and $(\beta,\theta_{\Sigma'}) \le 0$ for all $\beta \in \Delta(A(\Sigma,\Sigma'))$. Moreover, by equation (6.1), for any $\beta \in A(\Sigma,\Sigma')$, we have the following equivalences of conditions:

$$(\beta,\theta_\Sigma) < 0 \iff \beta \in N(u_{\Sigma,\Sigma'}) \iff (\beta,\theta_{\Sigma'}) < 0.$$

Lemma 6.2. *Let Σ and Σ' be distinct components of Π_0. Then*

$$u_{\Sigma,\Sigma'} \in \mathcal{W}_\sigma^{ab}.$$

Proof. By formula (6.1), for all $\beta \in N(u_{\Sigma,\Sigma'})$, $ht(\beta) > 0$. Therefore, it suffices to prove that, for all $\beta \in \Delta(A(\Sigma,\Sigma'))$, $ht_\sigma(\beta) < 2$. Assume by contradiction that $\beta \in \Delta(A(\Sigma,\Sigma'))$ and $ht_\sigma(\beta) < 2$. Then $ht_\sigma(k\delta-\beta) = 0$, hence $k\delta - \beta$ belongs to some component Σ'' of Π_0. At least one among Σ, Σ', say Σ, is not Σ''. Hence $(k\delta-\beta,\theta_\Sigma) = 0$, which gives $(\beta,\theta_\Sigma) = 0$: this is impossible, by Remark 6.1. □

Remark 6.2. If Z is a connected component of $A(\Sigma,\Sigma')$, then the sum of the roots in Z is a root and, by the Lemma 6.2, it has σ-height at most 1. This implies, in particular, that Z contains at most one root of Π_1.

Though we shall not need this fact, we notice that $A(\Sigma,\Sigma')$ is connected except in type $A_n^{(1)}$, in which case $A(\Sigma,\Sigma') = \Pi_1$, with Π_1 disconnected, since $\Sigma \neq \Sigma'$.

Lemma 6.3. *Let Σ and Σ' be distinct components of Π_0. If θ_Σ and $\theta_{\Sigma'}$ are both of type 1, then*

(1) *$u_{\Sigma,\Sigma'}(\theta_\Sigma) = \theta_{A(\Sigma')} = k\delta - \theta_{\Sigma'}$, and $u_{\Sigma,\Sigma'}$ is the element of minimal length in $\widehat{W}$, with this property;*
(2) *$u_{\Sigma,\Sigma'}^2 = 1$.*

Proof.
(1) Set $u = u_{\Sigma,\Sigma'}$. Since $L(u) \subset \Pi_1$, by Proposition 2.1 (3), it suffices to show that $u(\theta_\Sigma) = \theta_{A(\Sigma')} = k\delta - \theta_{\Sigma'}$. This is equivalent to show that $u^{-1}(\theta_{\Sigma'}) = k\delta - \theta_\Sigma$. Since $\theta_{\Sigma'}$ is of type 1, hence long, and $u^{-1}(\theta_{\Sigma'}) \in A(\Sigma)$, it suffices to show that $(u^{-1}(\theta_{\Sigma'}),\gamma) \ge 0$ for each $\gamma \in A(\Sigma) = A(\Sigma,\Sigma') \cup (\Sigma'\backslash\Gamma(\Sigma'))$. We know that $u = u_{0,\Pi_1}u_0$, where u_0 is the longest element of $W(A(\Sigma,\Sigma'))$ and u_{0,Π_1} is the longest element of $A(\Sigma,\Sigma')\backslash\Pi_1$. Since the only roots in $A(\Sigma,\Sigma')$ not orthogonal to $\theta_{\Sigma'}$ are the roots in Π_1, we see that $u^{-1}(\theta_{\Sigma'}) = u_0(\theta_{\Sigma'})$. Thus, since $(\theta_{\Sigma'},\gamma) \le 0$

when $\gamma \in A(\Sigma,\Sigma')$, we see that $(u^{-1}(\theta_{\Sigma'}),\gamma) = (\theta_{\Sigma'}, u_0(\gamma)) \geq 0$ for $\gamma \in A(\Sigma,\Sigma')$. Next we deal with the case $\gamma \in \Sigma' \setminus \Gamma(\Sigma')$. If $(\gamma,\theta_{\Sigma'}) = 0$ then $u(\gamma) = \gamma$, hence $(u^{-1}(\theta_{\Sigma'}),\gamma) = (\theta_{\Sigma'},\gamma) = 0$. If instead $(\gamma,\theta_{\Sigma'}) \neq 0$ and $ht_\sigma(u(\gamma)) = 1$, we are done because $(\theta^\vee_{\Sigma'}, u(\gamma)) = (\theta^\vee_{\Sigma'},\gamma) - 1 \geq 0$. If $ht_\sigma(u(\gamma)) = 2$ then $ht_\sigma(k\delta - u(\gamma)) = 0$, so $k\delta - u(\gamma)$ belongs to some component of Π_0. If this component is Σ', then $0 = (k\delta - u(\gamma), \theta_\Sigma)$ gives a contradiction, since $c_\eta(k\delta - u(\gamma)) \neq 0$ for all $\eta \in \Sigma$ such that $(\eta,\theta_\Sigma) \neq 0$. In the other case we have $0 = (k\delta - u(\gamma), \theta_{\Sigma'})$ hence $0 = (u(\gamma),\theta_{\Sigma'})$ and we are done.
(2) Set again $u = u_{\Sigma,\Sigma'}$. Since $u_0(\theta_\Sigma) = k\delta - \theta_{\Sigma'}$ we see that, if $\alpha \in \Pi_1$, then $u_0(\alpha) = -\alpha$. In fact, if Z is the component of $A(\Sigma,\Sigma')$ containing α, then, by Remark 6.2, α is the only root in Z that is not orthogonal to $\theta_{\Sigma'}$. By [6], it follows that u is an involution which permutes $A(\Sigma,\Sigma') \setminus \Pi_1$ and maps $\alpha \in \Pi_1$ to $-\theta_Z$. □

Lemma 6.4. *Let $\Sigma \neq \Sigma'$, $\theta_\Sigma, \theta_{\Sigma'}$ of type 1, $\alpha \in A(\Sigma)_{\overline{\ell}}$, $\beta \in A(\Sigma')_{\overline{\ell}}$, and assume that $w \in \mathcal{I}_{\alpha,k\delta-\theta_\Sigma} \cap \mathcal{I}_{\beta,k\delta-\theta_{\Sigma'}}$. Then*

1. *$u_{\Sigma,\Sigma'} \leq w$;*
2. *$\alpha \in \Sigma'$ and $\beta \in \Sigma$;*
3. *$uv_\alpha v_\beta \leq w$, where v_α is the element of minimal length in $W(\Sigma')$ that maps α to $\theta_{\Sigma'}$, and v_β is the element of minimal length in $W(\Sigma)$ that maps β to θ_Σ. Moreover, $uv_\alpha = w_{\alpha,k\delta-\theta_\Sigma}$, and $uv_\beta = w_{\beta,k\delta-\theta_{\Sigma'}}$.*

Proof.
(1). Let $u = u_{\Sigma,\Sigma'}$. If $u \not\leq w$, then there is $\gamma \in N(u)$ such that $\gamma \notin N(w)$. Note that $(\gamma,\theta^\vee_\Sigma) = (\gamma,\theta^\vee_{\Sigma'}) = -1$, hence $\theta_\Sigma + \gamma, \theta_\Sigma + \gamma + \theta_{\Sigma'} \in \widehat{\Delta}$. In particular we have that $k\delta - \theta_\Sigma - \gamma \in N(w)$. But then $k\delta - \theta_{\Sigma'} + k\delta - \gamma - \theta_\Sigma = 2k\delta - \theta_\Sigma - \gamma - \theta_{\Sigma'} \in N(w)$, which is absurd. We have therefore $u \leq w$.
(2)-(3). From (1) we obtain that $w = uv$ with $v(\alpha) = \theta_{\Sigma'}$. Let $U = \{\beta \in N(v) \mid \theta_{\Sigma'} - \beta \in \widehat{\Delta}^+\}$. Arguing as in the proof of Lemma 4.6, we see that U is biconvex, hence there is an element $x \in W(\Sigma')$ such that $N(x) = U$. Since x satisfies the hypothesis of Proposition 2.1 (6), we see that there is a root $\gamma \in \Sigma'$ such that $x = v_\gamma$, where v_γ is the element of minimal length that maps γ to $\theta_{\Sigma'}$. We conclude that $v_\gamma \leq v$. We now show that $\ell(uv_\gamma) = \ell(u) + \ell(v_\gamma)$; for this it suffices to prove that $u^{-1}(\eta) = u(\eta) \in \widehat{\Delta}^+$ for $\eta \in N(v_\gamma)$. If not, then $\eta \in N(u)$, hence, by Remark 6.1, $(\eta,\theta^\vee_{\Sigma'}) < 0$; but $\eta \in \Sigma'$, hence $(\eta,\theta_{\Sigma'}) \geq 0$. We now prove that $L(uv_\gamma) = \Pi_1$. We have $N(uv_\gamma) = N(u) \cup u(N(v_\gamma))$. Since $L(u) = \Pi_1$, it suffices to prove that $u(\eta) \notin \widehat{\Pi}$ for any $\eta \in N(v_\gamma)$. Since $\eta \in \Sigma'$, we have

$$(u(\eta),\theta_{\Sigma'}) = (\eta, u(\theta_{\Sigma'})) = (\eta, k\delta - \theta_\Sigma) = 0.$$

This implies that if $u(\eta) = \xi \in \widehat{\Pi}$, then $\xi \notin \Pi_1$ and, since $u \in W(A(\Sigma,\Sigma'))$, we see that, for any $\nu \in B_{\Sigma'}$, we have $0 = c_\nu(\xi) = c_\nu(u(\eta)) = c_\nu(\eta)$, hence $(\eta,\theta_{\Sigma'}) = 0$, against Proposition 2.1 (8). Since $uv_\gamma(\gamma) = k\delta - \theta_\Sigma = \theta_{A(\Sigma')}$ and $L(uv_\gamma) \subset \Pi_1$, we can apply Proposition 2.1 (3), to get $uv_\gamma = w_{\gamma,k\delta-\theta_\Sigma}$. This implies that $w_{\gamma,k\delta-\theta_\Sigma} \leq w$, so, by Proposition 5.1, $w \in \mathcal{I}_{\gamma,\mu}$, hence $\alpha = \gamma \in \Sigma'$ and $uv_\alpha \leq w$. Similarly, $\beta = \gamma \in \Sigma$ and $uv_\beta \leq w$. Since

$$N(uv_\alpha v_\beta) = N(u) \cup u(N(v_\alpha)) \cup u(N(v_\beta)), \tag{6.5}$$

we get that $uv_\alpha v_\beta \leq w$. □

Proposition 6.5. *Assume $\Sigma \neq \Sigma'$, $\theta_\Sigma, \theta_{\Sigma'}$ of type 1, $\alpha \in A(\Sigma)_{\overline{\ell}}$, and $\beta \in A(\Sigma')_{\overline{\ell}}$. Then $\mathcal{I}_{\alpha,k\delta-\theta_\Sigma} \cap \mathcal{I}_{\beta,k\delta-\theta_{\Sigma'}} \neq \emptyset$ if and only if $\alpha \in \Sigma'$ and $\beta \in \Sigma$. In this case,*

$$\min(\mathcal{I}_{\alpha,k\delta-\theta_\Sigma} \cap \mathcal{I}_{\beta,k\delta-\theta_{\Sigma'}}) = uv_\alpha v_\beta,$$

where v_α is the element of minimal length in $W(\Sigma')$ that maps α to $\theta_{\Sigma'}$, and v_β is the element of minimal length in $W(\Sigma)$ that maps β to θ_Σ.

Proof. We first prove that, if $\alpha \in \Sigma'$ and $\beta \in \Sigma$, then $uv_\alpha v_\beta \in \mathcal{I}_{\alpha,k\delta-\theta_\Sigma} \cap \mathcal{I}_{\beta,k\delta-\theta_{\Sigma'}}$. Indeed, it is clear that it suffices to prove that $uv_\alpha v_\beta \in \mathcal{W}^{ab}_\sigma$. As shown above $w_{\alpha,k\delta-\theta_\Sigma} = uv_\alpha$ and $w_{\beta,k\delta-\theta_{\Sigma'}} = uv_\beta$. From (6.5) we deduce that $N(uv_\alpha v_\beta) = N(w_{\alpha,k\delta-\theta_\Sigma}) \cup N(w_{\beta,k\delta-\theta_{\Sigma'}})$ hence $uv_\alpha v_\beta$ is a σ-minuscule element. The remaining statements follow from Lemma 6.4. □

Definition 6.2. *Let $\Sigma \neq \Sigma'$, θ_Σ and $\theta_{\Sigma'}$ of type 1. Consider $\alpha \in \Sigma'_{\overline{\ell}}$, $\beta \in \Sigma_{\overline{\ell}}$ and let v_α be the element of minimal length in $W(\Sigma')$ that maps α to $\theta_{\Sigma'}$ and v_β the element of minimal length in $W(\Sigma)$ that maps β to θ_Σ. Then we set*

$$w_{\alpha,\beta} = u_{\Sigma,\Sigma'} v_\alpha v_\beta.$$

Proposition 6.6. *Let $\Sigma \neq \Sigma'$, $\theta_\Sigma, \theta_{\Sigma'}$ of type 1, $\alpha \in \Sigma'_{\overline{\ell}}$ and $\beta \in \Sigma_{\overline{\ell}}$. Then*

$$w_{\alpha,\beta} = \sup\{\min \mathcal{I}_{\alpha,\mu}, \min \mathcal{I}_{\beta,\mu'}\}$$

and

$$w_{\alpha,\beta} x \in \mathcal{I}_{\alpha,k\delta-\theta_\Sigma} \cap \mathcal{I}_{\beta,k\delta-\theta_{\Sigma'}}$$

if and only if

$$x \in W((\widehat{\Pi}_\alpha \cap \widehat{\Pi}_\beta) \setminus \Pi_1) \backslash W(\widehat{\Pi}_\alpha \cap \widehat{\Pi}_\beta).$$

Proof. Since $N(uv_\alpha v_\beta) = N(w_{\alpha,k\delta-\theta_\Sigma}) \cup N(w_{\beta,k\delta-\theta_{\Sigma'}})$, it follows that

$$w_{\alpha,\beta} = \sup\{w_{\alpha,k\delta-\theta_\Sigma}, w_{\beta,k\delta-\theta_{\Sigma'}}\}.$$

Take $x \in W((\widehat{\Pi}_\alpha \cap \widehat{\Pi}_\beta) \setminus \Pi_1) \backslash W(\widehat{\Pi}_\alpha \cap \widehat{\Pi}_\beta)$. We now show that $w_{\alpha,\beta} x \in \mathcal{I}_{\alpha,k\delta-\theta_\Sigma} \cap \mathcal{I}_{\beta,k\delta-\theta_{\Sigma'}}$. We may assume that $x \neq 1$, in particular $|\Sigma| > 1$. It suffices to see that $w_{\alpha,\beta} x$ is σ-minuscule. Writing $w_{\alpha,\beta} x = w_{\alpha,k\delta-\theta_\Sigma} v_\beta x$, by the proof of Theorem 5.6, it suffices to prove that $v_\beta x \in V_{\alpha,k\delta-\theta_\Sigma}$. Since we already know that $v_\beta \in V_{\alpha,k\delta-\theta_\Sigma}$, we are left with proving that $ht_{B_\Sigma}(v_\beta(\gamma)) = 1$ for each $\gamma \in N(x)$. We have $(v_\beta(\gamma), \theta_\Sigma) = (\gamma, \beta) = 0$, hence $ht_{B_\Sigma}(v_\beta(\gamma)) = ht_\sigma(v_\beta(\gamma)) = ht_\sigma(\gamma) \geq 1$. Actually, the latter σ-height is 1: if it were 2, then $k\delta - \gamma$ would belong to some component, but this is impossible since both α and β belong to its support.

Vice versa, assume $w_{\alpha,\beta} x \in \mathcal{I}_{\alpha,k\delta-\theta_\Sigma} \cap \mathcal{I}_{\beta,k\delta-\theta_{\Sigma'}}$, with $\ell(w_{\alpha,\beta} x) = \ell(w_{\alpha,\beta}) + \ell(x)$ and $x \neq 1$. By Lemma 5.2, we get $v_\beta x \in \widehat{W}_\alpha, v_\alpha x \in \widehat{W}_\beta$; but $v_\beta \in \widehat{W}_\alpha$, hence $x \in \widehat{W}_\alpha$ and similarly $x \in \widehat{W}_\beta$. We are left with proving that $L(x) \subseteq \Pi_1$, so take $\gamma \in N(x) \cap \widehat{\Pi}$. Recall that $v_\beta x \in V_{\alpha,k\delta-\theta_\Sigma}$, hence

$$1 = ht_{B_\Sigma}(v_\beta(\gamma)) = ht_\sigma(v_\beta(\gamma)). \tag{6.6}$$

If $\gamma \notin \Pi_1$, then $v_\beta(\gamma) \in \widehat{\Delta}_0$, so $ht_\sigma(v_\beta(\gamma)) = 0$ against (6.6). Therefore $\gamma \in \Pi_1$, as desired. □

Proposition 6.7. *Assume $\mu, \mu' \in \mathcal{M}_\sigma$, and $\alpha, \beta \in \widehat{\Pi}$. Then $\mathcal{I}_{\alpha,\mu} \cap \mathcal{I}_{\beta,\mu'} \neq \emptyset$ if and only if either $\alpha = \beta$ and $\mu = \mu'$, or $\mu = k\delta - \theta_\Sigma$, $\mu' = k\delta - \theta'_\Sigma$, with $\Sigma \neq \Sigma'$ and θ_Σ, $\theta_{\Sigma'}$ of type 1, $\alpha \in \Sigma'_{\overline{\ell}}$, and $\beta \in \Sigma_{\overline{\ell}}$.*

Proof. In Proposition 6.5, we settled the cases $\mu = k\delta - \theta_\Sigma$, $\mu' = k\delta - \theta'_\Sigma$, with $\Sigma \neq \Sigma'$ and θ_Σ, $\theta_{\Sigma'}$ of type 1. It remains to prove that $\mathcal{I}_{\alpha,\mu} \cap \mathcal{I}_{\beta,\mu'} = \emptyset$ in all other non trivial cases.

We suppose by contradiction that there is $w \in \mathcal{I}_{\alpha,\mu} \cap \mathcal{I}_{\beta,\mu'}$ and treat the possibile cases one by one.

1. Let $\alpha, \beta \in \Pi_1$ and $\mu = k\delta + \beta$, $\mu' = k\delta + \alpha$. Since $N(w_{\beta,k\delta+\alpha}) \subset N(w)$ and $w^{-1}_{\beta,k\delta+\alpha}(\alpha) = -k\delta + \beta$ we see that $\alpha \in N(w)$. If $\Pi_0 = \emptyset$ then $(\alpha, \beta) \neq 0$, so $k\delta + \alpha + \beta \in \widehat{\Delta}$ and this implies that $k\delta + \alpha + \beta \in N(w)$. This is impossible since $ht_\sigma(k\delta + \alpha + \beta) = 4$. If $\Pi_0 \neq \emptyset$ and $\Sigma | \Pi_0$ then $\theta_\Sigma + \alpha \in \widehat{\Delta}$, so $k\delta - \theta_\Sigma - \alpha \in \widehat{\Delta}^+$. Since $k\delta - \alpha = \theta_\Sigma + k\delta - \theta_\Sigma - \alpha$, using the explicit expression for $w_{\beta,k\delta+\alpha}$ given in Proposition 4.9, we see that $\theta_\Sigma + \alpha = s_\alpha(\theta_\Sigma) \in N(w)$. Since $\alpha + \beta + \theta_\Sigma \in \widehat{\Delta}$, this implies that $(k\delta + \beta) + (\alpha + \theta_\Sigma) \in N(w)$ and again this gives a contradiction.

2. Let $\alpha, \gamma \in \Pi_1$, $\mu = k\delta + \gamma$, $\mu' = k\delta - \theta_\Sigma$. As above, we see that $\theta_\Sigma + \gamma \in N(w_{\alpha,k\delta+\gamma}) \subset N(w)$. But then $k\delta - \theta_\Sigma + \theta_\Sigma + \gamma = k\delta + \gamma \in N(w)$ and this is impossible.

3. Let $\mu = k\delta - \theta_\Sigma$, $\mu' = k\delta - \theta_{\Sigma'}$ with θ_Σ of type 2. We have clearly $\Sigma \neq \Sigma'$. Assume first θ_Σ complex. If $\delta - \theta_\Sigma$ is a simple root then $\widehat{\Pi} = \Sigma \cup \Pi_1$ contrary to the assumption that $\Sigma \neq \Sigma'$. Thus $\delta - \theta_\Sigma$ is not simple. We now rely on the explicit description of $w_{\alpha,\mu}$ given in Lemma 4.7. If $\gamma \in \Pi_1$, then $\gamma \in N(s_{\delta-\theta_\Sigma})$, hence $2\delta - 2\theta_\Sigma - \gamma \in N(s_{\delta-\theta_\Sigma}) \subset N(w_{\alpha,\mu}) \subset N(w)$. But then $(2\delta - \theta_{\Sigma'}) + (2\delta - 2\theta_\Sigma - \gamma) = 4\delta - \theta_{\Sigma'} - \gamma - \theta_\Sigma \in N(w)$ and this is not possible. It remains to check the case when θ_Σ is short compact. There is only a case when this occurs and Π_0 has more than one component, namely type $B_n^{(1)}$ with $\Pi_1 = \{\alpha_{n-1}\}$. By the explicit description of $w_{\alpha,\mu}$ given for this case in Lemma 4.4, we see that $\theta_{\Sigma'} \in N(w_{\alpha,\mu}) \subset N(w)$ and this gives clearly a contradiction. □

7. Maximal elements and dimension formulas

In this Section we give a parametrization of the maximal ideals in $\mathcal{W}_\sigma^{ab}$ and compute their dimension.

As a first step in our classification of maximal ideals, we determine which $\mathcal{I}_{\alpha,\mu}$ admits maximum. Let Π_1^1 denote the set of roots of type 1 in Π_1.

Proposition 7.1.

(1) *If θ_Σ is of type 1 (resp. type 2) and $\alpha \in \Gamma(\Sigma)_{\overline{\ell}} \cup \Pi_1^1$ (resp. $\alpha \in \Sigma_{\overline{\ell}}$) then $\mathcal{I}_{\alpha,k\delta-\theta_\Sigma}$ has maximum.*

(2) *Suppose that $\alpha \in \Sigma', \beta \in \Sigma$, and $\Sigma \neq \Sigma'$. If $\theta_\Sigma, \theta_{\Sigma'}, \alpha, \beta'$, are all roots of type 1, then $\mathcal{I}_{\alpha,k\delta-\theta_\Sigma} \cap \mathcal{I}_{\beta,k\delta-\theta_{\Sigma'}}$ has maximum.*

(3) *If $\Pi_1^1 = \Pi_1$ and $\alpha, \beta \in \Pi_1$, then $\mathcal{I}_{\alpha,\beta+k\delta}$ has maximum whenever it is nonempty.*

Proof. Recall that, by Theorem 5.6, $\mathcal{I}_{\alpha,\mu}$ is isomorphic to $\widehat{W}_{\alpha,\mu} \backslash \widehat{W}_\alpha$. The subgroup $\widehat{W}_{\alpha,\mu}$ is standard parabolic for any α and μ except when $\mu = k\delta - \theta_\Sigma$, θ_Σ of type 1, $|\Sigma| > 1$, and $\alpha \in A(\Sigma) \setminus (\Sigma \cup \Pi_1)$. The existence of the maximum in cases (1) and (3) follows now from subsection 2.4.2. The same applies to $\mathcal{I}_{\alpha,k\delta-\theta_{\Sigma'}} \cap \mathcal{I}_{\beta,k\delta-\theta_\Sigma}$, by Theorem 6.1. □

We already saw in Proposition 5.1 that, in many cases, the maximal elements of $\mathcal{I}_{\alpha,\mu}$ are maximal in $\mathcal{W}_\sigma^{ab}$. The next result deals with the remaining cases.

Proposition 7.2. *If $\Pi_1 = \Pi_1^1 = \{\alpha, \beta\}$ (with possibly $\alpha = \beta$), θ_Σ is of type 1, and $w \in \mathcal{I}_{\alpha,k\delta-\theta_\Sigma}$, then $w \le max(\mathcal{I}_{\alpha,k\delta+\beta})$.*

Proof. By Proposition 7.1, $\mathcal{I}_{\alpha,k\delta-\theta_\Sigma}$ has maximum. From subsection 2.4.2, we see that its maximum is $w_{max} = w_{\alpha,k\delta-\theta_\Sigma} w_{0,B_\Sigma} w_{0,\widehat{\Pi}_\alpha}$, where w_{0,B_Σ} is the longest element of $W(\widehat{\Pi}_\alpha \backslash B_\Sigma)$ and $w_{0,\widehat{\Pi}_\alpha}$ is the longest element of $\widehat{W}_\alpha$. Clearly there is a root $\alpha_\Sigma \in \Sigma$ such that $(\alpha_\Sigma, \alpha) \neq 0$, and we note that this root is necessarily unique, for, otherwise, $\Sigma \cup \{\alpha\}$ would contain a loop, and this is only possible in the adjoint case of type A_n. But in this case α is not of type 1.

We now show that $w_{0,B_\Sigma} w_{0,\widehat{\Pi}_\alpha}(\alpha_\Sigma) = \theta_\Sigma$. This is clear if $|\Sigma| = 1$ so we assume $|\Sigma| > 1$. Recall that $w_{0,\alpha}$ is the longest element of $W((\Pi_0)_\alpha)$. Let w_{B_Σ} be the longest element of $W((\Pi_0)_\alpha \backslash B_\Sigma)$. Obviously $N(w_{B_\Sigma} w_{0,\alpha}) \subset N(w_{0,B_\Sigma} w_{0,\widehat{\Pi}_\alpha})$ and we know that $w_{B_\Sigma} w_{0,\alpha}(\alpha_\Sigma) = \theta_\Sigma$. We show that $v(\alpha_\Sigma) = \theta_\Sigma$ for any v such that $w_{B_\Sigma} w_{0,\alpha} \le v \le w_{0,B_\Sigma} w_{0,\widehat{\Pi}_\alpha}$. This is proven by induction on $\ell(v) - \ell(w_{B_\Sigma} w_{0,\alpha})$. Assume that $v(\alpha_\Sigma) = \theta_\Sigma$ and $w_{B_\Sigma} w_{0,\alpha} \le v < vs_\gamma \le w_{0,B_\Sigma} w_{0,\widehat{\Pi}_\alpha}$ with $\gamma \in \widehat{\Pi}_\alpha$. We need to prove that $vs_\gamma(\alpha_\Sigma) = \theta_\Sigma$. Set $(\alpha_\Sigma, \gamma^\vee) = -r$ with $r \in \mathbb{Z}^+$. Then $vs_\gamma(\alpha_\Sigma) = \theta_\Sigma + rv(\gamma)$. Observe that $v \in V_{\alpha,k\delta-\theta_\Sigma}$, so $ht_{B_\Sigma}(v(\gamma)) = 1$. It follows that $ht_{B_\Sigma}(vs_\gamma(\alpha_\Sigma)) = 2 + r$. We claim that $ht_{B_\Sigma}(\nu) \le 2$ for any $\nu \in \Delta(\widehat{\Pi} \backslash \{\alpha\})$. Indeed this is obvious if $|\Pi_1| = 1$ and, in the hermitian symmetric case it follows from (5.1) and the observation that, in this case, $ht_\sigma(\nu) \le 1$. We conclude that $r = 0$ and $vs_\gamma(\alpha_\Sigma) = \theta_\Sigma$.

Having shown that $w_{0,B_\Sigma} w_{0,\widehat{\Pi}_\alpha}(\alpha_\Sigma) = \theta_\Sigma$, we have $w_{max}(\alpha_\Sigma) = w_{\alpha,k\delta-\theta_\Sigma}(\theta_\Sigma)$. Now $ht_\sigma(w_{\alpha,k\delta-\theta_\Sigma}(\theta_\Sigma)) = (w_{\alpha,k\delta-\theta_\Sigma}(\theta_\Sigma), k\delta - \theta_\Sigma^\vee) + ht_{B_\Sigma}(\theta_\Sigma) = 1$. This proves that $w_{max} s_{\alpha_\Sigma} \in \mathcal{W}_\sigma^{ab}$, so $w_{max} s_{\alpha_\Sigma} \in \mathcal{I}_{\alpha,k\delta+\beta}$ so $w_{max} \le max(\mathcal{I}_{\alpha,k\delta+\beta})$. □

Proposition 7.1 allows us to give the following definition:

Definition 7.1. *If θ_Σ is of type 1 (resp. type 2) and $\alpha \in \Gamma(\Sigma)_{\overline{\ell}}$ (resp. $\alpha \in \Sigma_{\overline{\ell}}$), we let $MI(\alpha)$ be the maximum of $\mathcal{I}_{\alpha,k\delta-\theta_\Sigma}$. If $\Sigma \neq \Sigma'$ and $\theta_\Sigma, \theta_{\Sigma'}, \alpha \in \Sigma', \beta \in \Sigma$ are all roots of type 1, we let $MI(\alpha, \beta)$ be the maximum of $\mathcal{I}_{\alpha,k\delta-\theta_{\Sigma'}} \cap \mathcal{I}_{\beta,k\delta-\theta_\Sigma}$. If $\alpha, \beta \in \Pi_1^1$ with $\mathcal{I}_{\alpha,k\delta+\beta} \neq \emptyset$, we let $MI(\alpha)$ be its maximum.*

We are finally ready to state the main result of the paper, which gives a complete parametrization of the set of maximal abelian $\mathfrak{b}^{\bar{0}}$-stable subspaces in $\mathcal{W}_\sigma^{ab}$. For notational reasons, it is convenient to fix an arbitrary total order $\prec$ on the components of Π_0.

Theorem 7.3. *The maximal $\mathfrak{b}^{\bar{0}}$-stable abelian subalgebras are parametrized by the set*

$$\mathcal{M} = \left(\bigcup_{\substack{\Sigma | \Pi_0 \\ \Sigma \text{ of type 1}}} \Gamma(\Sigma)_{\overline{\ell}} \right) \cup \left(\bigcup_{\substack{\Sigma | \Pi_0 \\ \Sigma \text{ of type 2}}} \Sigma_{\overline{\ell}} \right) \cup \left(\bigcup_{\substack{\Sigma, \Sigma' | \Pi_0,\, \Sigma \prec \Sigma' \\ \Sigma, \Sigma' \text{ of type 1}}} (\Sigma_{\overline{\ell}} \times \Sigma'_{\overline{\ell}}) \right) \cup \Pi_1^1. \tag{7.1}$$

Remark 7.1. In the adjoint case, there is just one component Σ in Π_0, which is the set of simple roots of $\mathfrak{g}$. In the r.h.s of (7.1) the only surviving term is $\Sigma_{\overline{\ell}}$, so $\mathcal{M}$ is the set of long simple roots of $\mathfrak{g}$. This parametrization has been first discovered by Panyushev and Röhrle [16], [17].

Now we begin to work in view of the proof of Theorem 7.3. We need to study the maximal elements of $\mathcal{I}_{\alpha,\mu}$. This is immediate when $\mathcal{I}_{\alpha,\mu}$ has maximum, more

delicate in the other cases. We also need to determine when a maximal element of $\mathcal{W}_\sigma^{ab}$ occurs in different $\mathcal{I}_{\alpha,\mu}$'s. The description of the intersections among different $\mathcal{I}_{\alpha,\mu}$'s given in Section 6 is the key to solve both problems. We start with the following

Lemma 7.4. *Assume $\Sigma \neq \Sigma'$. If $\theta_\Sigma, \theta_{\Sigma'}, \alpha \in \Sigma$, are all roots of type 1 and $w \in \mathcal{I}_{\alpha,k\delta-\theta_{\Sigma'}}$ is maximal, then there is $\eta \in \Sigma'$ such that $w(\eta) = k\delta - \theta_\Sigma$.*

Proof. Write $w = w_{\alpha,k\delta-\theta_{\Sigma'}}x$ with x maximal in $V_{\alpha,k\delta-\theta_{\Sigma'}}$. If $\Sigma' = \{\theta_{\Sigma'}\}$, then by Lemma 5.5 and Theorem 5.6, $x = 1$, so $w(\theta_{\Sigma'}) = u_{\Sigma,\Sigma'}v_\alpha(\theta_{\Sigma'}) = k\delta - \theta_\Sigma$.

If $|\Sigma'| > 1$, then by Definition 5.3, we have that $\widehat{W}_{\alpha,k\delta-\theta_{\Sigma'}} \neq \{1\}$. It follows that x cannot be the longest element of $W(\widehat{\Pi}_\alpha)$, hence there is a root γ in $\widehat{\Pi}_\alpha$ such that $x(\gamma) > 0$. Since x is maximal, then $ht_{B_{\Sigma'}}(x(\gamma)) \neq 1$, hence $ht_{B_{\Sigma'}}(x(\gamma)) \in \{0, 2\}$. Next we exclude that $ht_{B_{\Sigma'}}(x(\gamma)) = 0$ for all γ. We start with proving that if $ht_{B_{\Sigma'}}(x(\gamma)) = 0$, then $x(\gamma)$ is simple. Indeed, if $x(\gamma) - \beta \in \widehat{\Delta}_\alpha^+$ with $\beta \notin B_{\Sigma'}$, then, by convexity of $N(x)$, we have that $\beta \in N(x)$, contradicting the fact that $x \in V_{\alpha,k\delta-\theta_{\Sigma'}}$. If, for all roots γ in $\widehat{\Pi}_\alpha$ such that $x(\gamma) > 0$ we have that $x(\gamma) \in \widehat{\Pi}\backslash B_{\Sigma'}$, then, arguing as in Proposition 3.3, we see that $N(x)$ is the set of roots β in $\widehat{\Delta}_\alpha^+$ such that $ht_{B_{\Sigma'}}(\beta) > 0$. Since $(\theta_{\Sigma'}, \theta_{\Sigma'}^\vee) = 2$ and $|\Sigma'| > 1$, we see that this contradicts again the fact that $x \in V_{\alpha,k\delta-\theta_{\Sigma'}}$.

Therefore there is γ such that $ht_{B_{\Sigma'}}(x(\gamma)) = 2$. Then, arguing as above, we see that $x(\gamma)$ is minimal among the roots β such that $ht_{B_{\Sigma'}}(\beta) = 2$. By Lemma 5.5, we have that $x(\gamma) = \theta_{\Sigma'}$.

Arguing as in the proof of parts (2), (3) of Lemma 6.4, one checks that there is $\eta \in \Sigma'$ such that $v_\eta \leq x$. It follows that $w_{\alpha,k\delta-\theta_{\Sigma'}}v_\eta \leq w$. Since $w_{\alpha,k\delta-\theta_{\Sigma'}}v_\eta = u_{\Sigma,\Sigma'}v_\alpha v_\eta \in \mathcal{I}_{\eta,k\delta-\theta_\Sigma}$, by Proposition 5.1 we have $w \in \mathcal{I}_{\eta,k\delta-\theta_\Sigma}$ as desired. ☐

We are now ready to prove Theorem 7.3.

Proof of Theorem 7.3. Consider the map $MI : \mathcal{M} \to \mathcal{W}_\sigma^{ab}$ defined in Definition 7.1. Let MAX be the set of maximal abelian $\mathfrak{b}^{\bar{0}}$-stable subalgebras of $\mathfrak{g}^{\bar{1}}$. By Propositions 5.1 and 7.2, it is clear that $MI(m) \in MAX$ for any $m \in \mathcal{M}$. We next prove that $MI : \mathcal{M} \to MAX$ is bijective. First we show that $MI(\mathcal{M}) = MAX$. Let w be maximal. By Proposition 3.3 we have that w is maximal in $\mathcal{I}_{\alpha,\mu}$ for some $\mu \in \mathcal{M}_\sigma$. If $\alpha \in \Pi_1$ and it is of type 2, then μ is of type 2, hence $\mu = k\delta - \theta_\Sigma$ with θ_Σ of type 2, but this case is ruled out by Theorem 4.10. We can therefore assume α of type 1. From Proposition 7.2 we deduce $\mu = \beta + k\delta$ so that $\alpha, \beta \in \Pi_1^1$. Hence $w = MI(\alpha)$. If $\alpha \notin \Pi_1$ then, by Proposition 4.9, we have that $\mu = k\delta - \theta_\Sigma$. If $\alpha \in \Sigma$ and θ_Σ is of type 1 (resp. type 2), then by Theorem 4.10, we have $\alpha \in \Gamma(\Sigma)_{\overline{\ell}}$ (resp. $\Sigma_{\overline{\ell}}$) and by Proposition 7.1 we have $w = MI(\alpha)$. Finally assume $\alpha \in \Sigma' \neq \Sigma$. In particular, by Theorem 4.10, α, θ_Σ, and $\theta_{\Sigma'}$ are of type 1. By Lemma 7.4 and Proposition 7.1 (2), we see that there is $\beta \in \Sigma'$ such that $w = MI(\alpha, \beta)$.

Finally we prove that MI is injective. Set

$$Y = \bigcup_{\substack{\Sigma|\Pi_0 \\ \Sigma \text{ of type } 1}} \Gamma(\Sigma)_{\overline{\ell}} \cup \bigcup_{\substack{\Sigma|\Pi_0 \\ \Sigma \text{ of type } 2}} \Sigma_{\overline{\ell}} \cup \Pi_1^1. \tag{7.2}$$

If $\alpha, \beta \in Y$, it follows readily from Theorem 6.1 that $MI(\alpha) = MI(\beta)$ implies $\alpha = \beta$. Theorem 6.1 also implies that $MI(\alpha) \neq MI(\beta, \gamma)$ for $\alpha \in Y$ and $(\beta, \gamma) \in \Sigma_{\overline{\ell}} \times \Sigma'_{\overline{\ell}}$ with $\beta, \gamma, \Sigma, \Sigma'$ of type 1. Suppose finally that $MI(\alpha, \beta) = MI(\gamma, \eta)$ with $\alpha \in \Sigma_{\overline{\ell}}$,

$\beta \in \Sigma'_{\overline{\ell}}$, $\gamma \in \Sigma''_{\overline{\ell}}$, $\eta \in \Sigma'''_{\overline{\ell}}$, and $\Sigma, \Sigma', \Sigma'', \Sigma'''$ all of type 1, and $\Sigma \prec \Sigma'$, $\Sigma'' \prec \Sigma'''$. Set $w = MI(\alpha, \beta) = MI(\gamma, \eta)$. We have $w \in \mathcal{I}_{\alpha,k\delta-\theta_{\Sigma'}} \cap \mathcal{I}_{\gamma,k\delta-\theta_{\Sigma'''}} \neq \emptyset$. Thus either $\alpha = \gamma$ and $\Sigma' = \Sigma'''$ or $\gamma \in \Sigma'$ and $\alpha \in \Sigma'''$. In the first case we have $w(\eta) = k\delta - \theta_\Sigma$ so $\beta = \eta$. In the second case we have $\Sigma = \Sigma'''$ and $\Sigma'' = \Sigma'$ contradicting the fact that $\Sigma'' \prec \Sigma'''$. □

As a complement to Theorem 7.3, we compute the dimension of maximal abelian subspaces.

Recall from (2.5) that g_R denotes the dual Coxeter number of a finite irreducible root system R. Suppose Σ is a component of Π_0. To simplify notation, we set $g_\Sigma = g_{\Delta(\Sigma)}$ and, if θ_Σ is type 1, $g_{A(\Sigma)} = g_{\Delta(A(\Sigma))}$ (note that in this case $\Delta(A(\Sigma))$ is irreducible by Remark 4.1). Also recall from Section 2.1 that K is the canonical central element of $\widehat{L}(\mathfrak{g}, \sigma)$ and $\mathbf{g}$ is its dual Coxeter number and from Section 3 that we denote by a the squared length of a long root in $\widehat{\Delta}^+$.

Lemma 7.5. *Let $\gamma \in \widehat{\Delta}_{re}$. Then*

1. *$(k\delta + \gamma)^\vee = \frac{a}{\|\gamma\|^2} K + \gamma^\vee$. In particular, $(k\delta - \theta_\Sigma)^\vee = r_\Sigma K - \theta_\Sigma^\vee$.*
2. *If θ_Σ is of type 1, then $g_{A(\Sigma)} = \mathbf{g} - g_\Sigma + 2$. In particular, if $\alpha \in A(\Sigma)_{\overline{\ell}}$, then $\ell(w_{\alpha,k\delta-\theta_\Sigma}) = \mathbf{g} - g_\Sigma$.*
3. *If θ_Σ is of type 2 and $\alpha \in \Sigma_{\overline{\ell}}$, then $\ell(w_{\alpha,k\delta-\theta_\Sigma}) = \mathbf{g} - 1$.*
4. *If $\alpha, \beta \in \Pi_1^1$, and $\beta \neq \alpha$ if $|\Pi_1^1| = 2$, then $\ell(w_{\beta,k\delta+\alpha}) = \mathbf{g} - 1$.*

Proof. We compute, using (2.1):

$$(k\delta + \gamma)^\vee = \frac{2k}{\|\gamma\|^2}\nu^{-1}(\delta) + \gamma^\vee = \frac{k}{a_0}\frac{\|\delta - a_0\alpha_0\|^2}{\|\gamma\|^2} K + \gamma^\vee.$$

A direct inspection shows that $\frac{k\|\delta - a_0\alpha_0\|^2}{a_0} = a$. This proves (1).

To prove the first part of (2) we observe that $g_{A(\Sigma)} = ht_{\widehat{\Pi}^\vee}((k\delta - \theta_\Sigma)^\vee) + 1$. The result then follows readily from (1). By Proposition 2.1 (4) and (4.5), we see that if θ_Σ is of type 1 and $\alpha \in A(\Sigma)_{\overline{\ell}}$ then $\ell(w_{\alpha,k\delta-\theta_\Sigma}) = g_{A(\Sigma)} - 2 = \mathbf{g} - g_\Sigma$. For (3) recall that $w_{\alpha,k\delta-\theta_\Sigma} = sv_\alpha$, s being the element of $\widehat{W}$ described in Lemma 4.3 and v_α the element of minimal length in $W(\Sigma)$ mapping α to θ_Σ. It follows that $\ell(w_{\alpha,k\delta-\theta_\Sigma}) = \ell(s) + g_\Sigma - 2$. It is therefore enough to show that $\ell(s) = \mathbf{g} - g_\Sigma + 1$. Start from the following formula, which is a variation of e.g. [10, Exercise 3.12]. It is easily proved by induction on $\ell(w)$:

$$w^{-1}(\lambda) = \lambda - \sum_{i=1}^{l} (\lambda, \beta_j^\vee)\alpha_{i_j}. \tag{7.3}$$

Here $w \in \widehat{W}$, $\lambda \in \widehat{\mathfrak{h}}^*$, $s_{i_1} \cdots s_{i_l}$ is a reduced expression of w and $\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$ (so that $N(w) = \{\beta_1, \dots, \beta_l\}$ and $l = \ell(w)$). Applying (7.3) to $w = s$ and $\lambda = k\delta - \theta_\Sigma$ and using Lemma 4.3, we obtain that $s(\lambda) = \lambda - 2\sum_{i=1}^{l} r_j\alpha_{i_j}$, where $r_j = \frac{\|\lambda\|^2}{\|\beta_j\|^2}$. In turn, recalling that $s(\mu) = \theta_\Sigma$ and applying $\frac{2}{(\theta_\Sigma, \theta_\Sigma)}\nu^{-1}$ to the previous equality we get

$$\theta_\Sigma^\vee = (k\delta - \theta_\Sigma)^\vee - 2\sum_{j=1}^{l} \alpha_{i_j}^\vee.$$

In particular, taking $ht_{\widehat{\Pi}^\vee}$ of both sides, we obtain $2\ell(s) = ht_{\widehat{\Pi}^\vee}((k\delta-\theta_\Sigma)^\vee) - g_\Sigma + 1$. Now use part (1) (recall that $r_\Sigma = 2$) to finish the proof.

To prove (4), we recall that, by Proposition 4.9 $w_{\beta,\alpha+k\delta} = s_\alpha w_{0,\alpha} w_0$, hence $N(w_{\beta,\alpha+k\delta}) = \{\alpha\} \cup s_\alpha N(w_{0,\alpha}w_0)$. By definition, for all $\gamma \in N(w_{0,\alpha}w_0)$, we have $(\gamma, \alpha^\vee) < 0$, hence $(s_\alpha\gamma, \alpha^\vee) > 0$. Now it is clear that $s_\alpha\gamma \neq \alpha$, so that $s_\alpha\gamma - \alpha$ is a root. Since

$$\frac{\|s_\alpha\gamma - \alpha\|^2}{\|\alpha\|^2} = 1 + \frac{\|s_\alpha\gamma\|^2}{\|\alpha\|^2} - (s_\alpha\gamma, \alpha^\vee).$$

and α is long, then $(s_\alpha\gamma, \alpha^\vee) = 2$ and $\|s_\alpha\gamma - \alpha\| = 0$ or $(s_\alpha\gamma, \alpha^\vee) = 1$. The first case implies $s_\alpha\gamma = c\delta + \alpha$ for some $c \in \mathbb{R} \setminus \{0\}$. This is not possible, since $ht_\sigma(s_\alpha\gamma) = 1$. Hence $(s_\alpha\gamma, \alpha^\vee) = 1$ for all $\gamma \in N(w_{0,\alpha}w_0)$. Now, formula (7.3) with $w = w_{\beta,\alpha+k\delta}$ and $\lambda = \alpha + k\delta$ gives $\beta = \alpha + k\delta - 2\alpha - \sum_{i=2}^{l}(\alpha, \beta_i^\vee)\alpha_{i_j}$, with $\{\beta_2, \dots, \beta_l\} = s_\alpha N(w_{0,\alpha}w_0)$ and, applying $\frac{2}{(\alpha,\alpha)}\nu^{-1}$,

$$\beta^\vee = K - \alpha^\vee - \sum_{i=2}^{l} \alpha_{i_j}^\vee.$$

It follows that $l = \mathbf{g} - 1$, as claimed. □

If $\mathfrak{g}$ is a simple Lie algebra, let $g_\mathfrak{g}$ be the dual Coxeter number of the root system of $\mathfrak{g}$. It is know that $\mathbf{g} = g_\mathfrak{g}$ if $\mathfrak{g}$ is simple and that $\mathbf{g} = g_\mathfrak{k}$ in the adjoint case $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}$. The following result gives our dimension formulas.

Theorem 7.6. *If θ_Σ is of type 1 and $\alpha \in \Gamma(\Sigma)_{\overline{\ell}}$, then*

$$\dim MI(\alpha) = \mathbf{g} - g_\Sigma + |\widehat{\Delta}^+_\alpha| - |\Delta^+(\widehat{\Pi}_{\alpha,\mu})|. \tag{7.4}$$

If $\alpha \in \Pi_1^1$, or $\alpha \in \Sigma_{\overline{\ell}}$ with θ_Σ of type 2, then

$$\dim MI(\alpha) = \mathbf{g} - 1 + |\widehat{\Delta}^+_\alpha| - |\Delta^+(\widehat{\Pi}_{\alpha,\mu})|. \tag{7.5}$$

If $\alpha \in \Sigma_{\overline{\ell}}$, $\beta \in \Sigma'_{\overline{\ell}}$, with $\Sigma \neq \Sigma'$ and θ_Σ, $\theta_{\Sigma'}$ of type 1, then

$$\dim MI(\alpha, \beta) = \mathbf{g} - 2 + |\Delta^+(\widehat{\Pi}_\alpha \cap \widehat{\Pi}_\beta)| - |\Delta^+((\widehat{\Pi}_\alpha \cap \widehat{\Pi}_\beta) \setminus \Pi_1)|. \tag{7.6}$$

Proof. By (2.8), Theorems 4.10 and 5.6 imply that, for $\alpha \in Y$ (cf. (7.2))

$$\dim MI(\alpha) = \ell(w_{\alpha,\mu}) + |\widehat{\Delta}^+_\alpha| - |\Delta^+(\widehat{\Pi}^*_{\alpha,\mu})|.$$

Note that in the current setting we have that $\widehat{\Pi}^*_{\alpha,\mu} = \widehat{\Pi}_{\alpha,\mu}$. Using part (2) of the previous Lemma we obtain (7.4). Likewise, if θ_Σ is of type 2 and $\alpha \in \Sigma_{\overline{\ell}}$, or $\alpha \in \Pi_1^1$ then (7.5) follows from (3), (4) in Lemma 7.5.

Finally, we have to prove (7.6). Theorem 6.1 gives

$$\dim MI(\alpha, \beta) = \ell(w_{\alpha,\beta}) + |\Delta^+(\widehat{\Pi}_\alpha \cap \widehat{\Pi}_\beta)| - |\Delta^+((\widehat{\Pi}_\alpha \cap \widehat{\Pi}_\beta) \setminus \Pi_1)|.$$

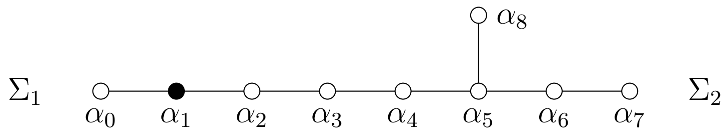
So it remains to show that $\ell(w_{\alpha,\beta}) = \mathbf{g} - 2$. From Lemma 6.4 (3), we know that $w_{\alpha,\beta} = u_{\Sigma,\Sigma'} v_\alpha v_\beta$ with $\ell(w_{\alpha,\beta}) = \ell(u_{\Sigma,\Sigma'}) + \ell(v_\alpha) + \ell(v_\beta)$, where v_α, v_β are the elements of minimal length mapping α, β, respectively, to the highest root of their component. By Proposition 2.1 (4), the lengths of the latter elements are $g_\Sigma - 2$, $g_{\Sigma'} - 2$, respectively. We know that $u_{\Sigma,\Sigma'} v_\beta$ is the element of minimal length in $W(A(\Sigma))$ mapping β to $k\delta - \theta_\Sigma$. Hence $\ell(u_{\Sigma,\Sigma'}) + \ell(v_\beta) = g_{A(\Sigma)} - 2$. Using Lemma 7.5 (1), we have

$$\ell(u_{\Sigma,\Sigma'}) = g_{A(\Sigma)} - g_{\Sigma'} = \mathbf{g} - g_\Sigma - g_{\Sigma'} + 2,$$

hence (7.6) is proven. □

Remark 7.2. The dimension formula in the adjoint case is a specialization of (7.5) and is due to Suter [19]. For a refinement of Suter's formula, see [2, Theorem 8.13].

Example 7.1. We illustrate our results when $\mathfrak{g}$ is of type E_8 and $\mathfrak{g}^{\bar 0}$ of type $A_1 \times E_7$. We denote by Σ_1 the component of Π_0 of type A_1 and by Σ_2 that of type E_7.



Then $A(\Sigma_1) = \widehat{\Pi} \setminus \{\alpha_0\}, \Gamma(\Sigma_1) = \emptyset$, $A(\Sigma_2) = \widehat{\Pi} \setminus \{\alpha_7\}, \Gamma(\Sigma_2) = \Sigma_2 \setminus \{\alpha_7\}$. Set

$$\mu_1 = -\theta_{\Sigma_1} + \delta, \quad \mu_2 = -\theta_{\Sigma_2} + \delta, \quad \mu_3 = \alpha_1 + \delta.$$

By Theorem 4.10, the poset $\mathcal{I}_{\alpha_i,\mu_j}$ is non empty if and only if

$$(i,j) \in \{(k,1) \mid 1 \le k \le 8\} \cup \{(k,2) \mid k = 0,1,2,3,4,5,6,8\} \cup \{(1,3)\}.$$

By Theorem 7.3, the maximal $\mathfrak{b}^{\bar 0}$-stable abelian subspaces are $MI(\alpha_i)$, $2 \le i \le 6$ and $i = 8$, $MI(\alpha_i, \alpha_0)$, $2 \le i \le 8$, $MI(\alpha_1)$. More explicitly,

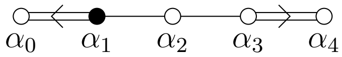
$$\begin{aligned} MI(\alpha_1) &= \max \mathcal{I}_{\alpha_1,\mu_3}, \\ MI(\alpha_i) &= \max \mathcal{I}_{\alpha_i,\mu_2},\ 2 \le i \le 8, i \ne 7, \\ MI(\alpha_i, \alpha_0) &= \max(\mathcal{I}_{\alpha_0,\mu_2} \cap \mathcal{I}_{\alpha_i,\mu_1}),\ 2 \le i \le 8. \end{aligned}$$

The following table displays the relevant data necessary to calculate the corresponding dimensions using the formulas provided in Theorem 7.6. The formula in question appears in the leftmost column. Recall that $\mathbf{g} = 30$ and that $g_{E_7} = 18$. It is easily checked that $B_{\mu_2} = \{\alpha_7\}, B_{\mu_3} = \{\alpha_1\}$.

(7.4)	$MI(\alpha)$	type of $\widehat{\Delta}_\alpha$	type of $\Delta(\widehat{\Pi}_{\alpha,\mu})$	$\dim MI(\alpha)$
	$MI(\alpha_6)$	$A_5 \times A_1$	$A_5 \times A_1$	30-18+16-16=12
	$MI(\alpha_5)$	$A_4 \times A_1$	A_4	30-18+11-10=13
	$MI(\alpha_4)$	$A_3 \times A_2 \times A_1$	$A_3 \times A_1 \times A_1$	30-18+10-8=14
	$MI(\alpha_8)$	$A_5 \times A_2$	$A_5 \times A_1$	30-12+18-16=14
	$MI(\alpha_3)$	$A_2 \times A_4$	$A_2 \times A_3$	30-18+13-9=16
	$MI(\alpha_2)$	$A_1 \times D_5$	$A_1 \times D_4$	30-18+21-13=20
(7.6)	$MI(\alpha,\beta)$	type of $\Delta(\widehat{\Pi}_\alpha \cap \widehat{\Pi}_\beta)$	type of $\Delta((\widehat{\Pi}_\alpha \cap \widehat{\Pi}_\beta) \setminus \Pi_1)$	$\dim MI(\alpha)$
	$MI(\alpha_0,\alpha_i)$, $2 \le i \le 8$	R	R	28=30-2
(7.5)	$MI(\alpha)$	type of $\widehat{\Delta}_\alpha$	type of $\Delta(\widehat{\Pi}_{\alpha,\mu})$	$\dim MI(\alpha)$
	$MI(\alpha_1)$	E_6	E_6	29=30-1+36-36

The symbol R in the fourth to last line of the previous table means that if $\Delta(\widehat{\Pi}_\alpha \cap \widehat{\Pi}_\beta)$ is of type R, then also $\Delta((\widehat{\Pi}_\alpha \cap \widehat{\Pi}_\beta) \setminus \Pi_1)$ is of type R. This happens because $\alpha_1 \notin \widehat{\Pi}_{\alpha_0}$.

Example 7.2. Here we illustrate another example corresponding to $\mathfrak{g}$ of type D_5 and $\mathfrak{g}^{\bar 0}$ of type $A_1 \times B_3$. We denote by Σ_1 the component of Π_0 of type A_1 and by Σ_2 that of type B_3. The corresponding picture is

Σ_1  Σ_2

Then $A(\Sigma_2) = \{\alpha_0, \alpha_1, \alpha_2\}, \Gamma(\Sigma_2) = \{\alpha_2\}$. Set

$$\mu_1 = -\theta_{\Sigma_1} + 2\delta = -\alpha_0 + 2\delta, \quad \mu_2 = -\theta_{\Sigma_2} + 2\delta, \quad \mu_3 = \alpha_1 + 2\delta.$$

By Theorem 4.10, the poset $\mathcal{I}_{\alpha_i,\mu_j}$ is non empty if and only if

$$(i, j) \in \{(0, 1), (1, 2), (1, 3), (2, 2)\}.$$

By Theorem 7.3, the maximal $\mathfrak{b}^{\bar 0}$-stable abelian subspaces are $MI(\alpha_i)$, $0 \le i \le 2$. Recall that $\mathbf{g} = 8$ and that $g_{B_3} = 5$. It is easily checked that $B_{\mu_1} = B_{\mu_3} = \{\alpha_1\}, B_{\mu_2} = \{\alpha_3\}$.
Since $\alpha_0 = \theta_{\Sigma_1}$ is of type 2, we can calculate $\dim(MI(\alpha_0))$ using formula (7.5). Since $\Delta(\widehat{\Pi}_{\alpha_0,\mu_1}) = \widehat{\Delta}_{\alpha_0}$, we obtain

$$\dim(MI(\alpha_0)) = 8 - 1 = 7.$$

Exactly the same calculation works for $\alpha = \alpha_1$, so that $\dim(MI(\alpha_1)) = 7$. Finally $\dim(MI(\alpha_2))$ is computed by (7.4). We have that $\widehat{\Pi}_{\alpha_2} = \{\alpha_0, \alpha_4\}$, so $\widehat{\Delta}_{\alpha_2}$ is of type $A_1 \times A_1$ and $\widehat{\Pi}_{\alpha_2,\mu_2} = \widehat{\Pi}_{\alpha_2}$. Thus

$$\dim(MI(\alpha_2)) = 8 - 5 + 2 - 2 = 3.$$

Proposition 7.7. *In the hermitian case, if $\alpha \in \Pi_1$, we have*

$$\dim(MI(\alpha)) = \frac{\dim(\mathfrak{g}^{\bar 1})}{2}.$$

Proof. Let $\Pi_1 = \{\alpha, \beta\}$. It is clear that a root of $\widehat{L}(\mathfrak{g}, \sigma)$ has σ-height 1 if it is greater or equal than exactly one among α, β. Hence

$$t^{-1} \otimes \mathfrak{g}^{\bar 1} = \bigoplus_{\substack{\gamma \ge \alpha \\ \beta \notin Supp(\gamma)}} \widehat{L}(\mathfrak{g}, \sigma)_{-\gamma} \oplus \bigoplus_{\substack{\gamma \ge \beta \\ \alpha \notin Supp(\gamma)}} \widehat{L}(\mathfrak{g}, \sigma)_{-\gamma}. \tag{7.7}$$

Since there is an automorphism of the Dynkin diagram of $\widehat{L}(\mathfrak{g}, \sigma)$ switching the elements of Π_1, the two summands in the r.h.s. of (7.7) have both dimension $\dim(\mathfrak{g}^{\bar 1})/2$. Set $F_\alpha = \{-\gamma \in \widehat{\Delta} \mid \gamma \ge \beta$ *and* $\alpha \notin$ *Supp*$(\gamma)\}$. It is clear that, if $-\gamma', -\gamma'' \in F_\alpha$, then $-\gamma' - \gamma'' \notin \widehat{\Delta}$; moreover, for each $\eta \in \Delta_0^+$ such that $-\gamma + \eta \in \widehat{\Delta}$ we have that $-\gamma + \eta \in F_\alpha$. It follows that $\bigoplus_{\substack{\gamma \ge \beta \\ \alpha \notin Supp(\gamma)}} \widehat{L}(\mathfrak{g}, \sigma)_{-\gamma}$ is an abelian $\mathfrak{b}^{\bar 0}$-stable subspace of $t^{-1} \otimes \mathfrak{g}^{\bar 1}$, hence, by Remark 3.6, it corresponds to a $\mathfrak{b}^{\bar 0}$-stable abelian subspace of $\mathfrak{g}^{\bar 1}$. In order to conclude the proof, we shall prove that the element of $\mathcal{W}_\sigma^{ab}$ corresponding to the latter subspace is $MI(\alpha)$. Set $z = MI(\alpha)$. By formula (4.5), Theorem 4.10, and Lemma 5.2, $z = s_\beta z'$ with $z' \in W(\widehat{\Pi} \setminus \{\alpha\})$. It follows that $N(z) \subseteq -F_\alpha$ and therefore, by the maximality of $MI(\alpha)$, that $N(z) = -F_\alpha$: this proves the claim. □

Remark 7.3. If we take $\Pi_1 = \{\alpha_0, \beta\}$, where α_0 is the extra node of the extended Dynkin diagram associated to $\mathfrak{g}$, then the sum $\mathfrak{i}$ of all root subspaces corresponding to $\{\gamma \geq \beta \mid \alpha_0 \notin Supp(\gamma)\}$ is an ideal of the Borel subalgebra of $\mathfrak{g}$ corresponding to the simple system $\widehat{\Pi} \setminus \{\alpha_0\}$. Moreover, if w is the element associated to this abelian ideal via Peterson's bijection quoted in the Introduction, then $N(w) = \{\gamma \geq \alpha_0 \mid \beta \notin Supp(\gamma)\}$. Now Proposition 4.9 implies that $w(\beta) = \delta + \alpha_0$, hence this ideal is included in the maximal ideal associated to β via the Panyushev bijection [17]. By Theorem 7.6 and Suter dimension formula, we obtain that $\mathfrak{i}$ is exactly this maximal ideal. Notice that this applies to any simple root β of $\mathfrak{g}$ that occurs with coefficient 1 in the highest root of $\mathfrak{g}$.

Remark 7.4. As recalled in the Introduction, Panyushev [15] investigated the maximal eigenvalue of the Casimir element of $\mathfrak{g}^{\bar{0}}$ w.r.t. the Killing form of $\mathfrak{g}$. In particular he showed that in the hermitian case $N = \frac{\dim(\mathfrak{g}^{\bar{1}})}{2}$ gives the required maximal eigenvalue. By the previous Proposition, if $v_1, \dots, v_N$ is any basis of $MI(\alpha)$, then $v_1 \wedge \dots \wedge v_N$ is an explicit eigenvector of maximal eigenvalue.

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